Spectral Sequences and the Homology of Affine Groups

by

Ben Windsor

MA4K9 Dissertation

Submitted to The University of Warwick

Mathematics Institute

March, 2018

Contents

1 Introduction

Introduced by Jean Leray around the second world war, spectral sequences have become a ubiquitous algebraic tool. They were first used to aid in the computation of sheaf cohomology, but it was then realised they could be applied to many other algebraic problems, especially in topology. In the following years spectral sequences were formalised, through the languages of homological algebra and category theory, to become an essential part of many mathematicians toolbox.

It was around a similar time that group homology was being formally defined. Following from the homology of spaces, mathematicians worked to see if they could extract information about other objects with similar constructions. This lead to the creation of group cohomology and homology, where spectral sequences became a vital calculation tool.

This projects main aim is to be able to work with spectral sequences and group homology in the context of a recent paper 'Euler class groups and the homology of elementary and special linear groups' by M.Schlichting [SCH]. In our work with this paper we will re-prove one of its results with an alternate method, and use the proof techniques of Schlichting to re-prove an older result that was essential in the study of homology stability and K-theory. I want to guide the reader through the subjects at hand, giving detail and examples where needed and culminating in our work with Schlichting's paper.

1.1 Assumed Knowledge

We assume the reader has knowledge of some basic category theoretic terminology (basic definitions, see [AWO]), and introductory homological algebra (see [ROT] and [WEI]) including knowledge of projective modules, chain complexes, tensor products, and the homology of a chain complex. No knowledge of spectral sequences or group homology is assumed, though our definition of group homology is only brief. For group homology we follow [BRO] and for spectral sequences we introduce them following [ROT] and then later use [BRO] and [KNU].

2 Category Theory and Homological Algebra

We start with some categorical and homological constructs and worked examples. This chapter forms the foundational material for the later chapters.

2.1 Derived Functors

[ROT, §6.2] Given two categories $\mathcal{C}$ and $\mathcal{D}$, with enough projectives and a right exact covariant functor $F : \mathcal{C} \to \mathcal{D}$ we can construct what we call the left derived functors of F. Derived functors play an integral role in many areas of mathematics, and allow us to define group homology (and cohomology).

Definition 1. *For an object $X \in \mathcal{C}$, and a projective resolution $P_\bullet \to X$, apply F to $P_\bullet$ to obtain the chain complex*

$$\cdots \to FP_1 \to FP_0 \to 0.$$

Denote the n'th homology of this complex as $L^nF(X)$ and call this the n'th left derived functor of F applied to X.

We can do this for any object in $\mathcal{C}$, and we know a morphism $f : X \to Y$ will extend to a map of projective resolutions (proved analogous to §3.3 lemma 1, or [ROT, Thm. 6.16]) and so we have our left derived functor $L^nF : \mathcal{C} \to \mathcal{D}$. In an analogous way we can also define right derived functors R^nF of a left exact functor.

In particular if we work over a ring R and fix a left R-module M, we get the covariant right exact functor $F := - \otimes_R M$, which maps from right R-modules to right R-modules. Similarly can get the functor contravariant left exact functor $G := \mathrm{Hom}_R(-, M)$.

Definition 2. *We denote the specific derived functors of these functors as $\mathrm{Tor}_n^R(N, M) := L^nF(N)$ and $\mathrm{Ext}_R^n(N, M) := R^nG(N)$.*

Note that there are functors $N \otimes_R -$ and $\mathrm{Hom}_R(N, -)$ and taking a different type of resolution we can arrive at the same derived functors Tor and Ext.

[ROT, Thm. 6.27] Right and left derived functors give useful long exact sequences, similar to the long exact sequence associated to a short exact sequence of chain complexes. Consider a short exact sequence of modules

$$0 \to A \to B \to C \to 0$$

then for a functor F as above, we have a long exact sequence

$$\cdots \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0,$$

and when F gives a right derived functor we get

$$0 \to F(A) \to F(B) \to F(C) \to R_1F(A) \to R_1F(B) \to R_1F(C) \to \cdots ,$$

which can be useful tools for calculation.

[ROT, Thm. 7.2, Corollary 7.25] These functors give alternate definitions of projective, injective and flat modules. We can define a projective module P as one that makes

$\mathrm{Hom}(P, -)$ an exact functor, i.e. $\mathrm{Ext}^n_R(P, -) = 0$ for all $n > 0$. Similarly we can define injective modules using $\mathrm{Hom}(-, P)$ and flat modules with the exactness of $P \otimes -$ and hence the vanishing of all $\mathrm{Tor}_n(P, -)$ for $n > 0$.

2.2 Calculating with Tor and Ext

So to see some computations with these functors lets first consider the simplest case $\mathrm{Tor}^R_0(N, M)$ and $\mathrm{Ext}^0_R(N, M)$. Using the canonical free resolution of N

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0,$$

we have an exact sequence

$$F_1 \otimes_R M \xrightarrow{f_1 \otimes \mathrm{id}} F_0 \otimes_R M \xrightarrow{f_0 \otimes \mathrm{id}} N \otimes_R M \to 0.$$

So by the first isomorphism theorem we obtain

$$N \otimes_R M \cong \frac{F_0 \otimes_R M}{\mathrm{Ker}(f_0 \otimes \mathrm{id})} \cong \frac{F_0 \otimes_R M}{\mathrm{Im}(f_1 \otimes \mathrm{id})}.$$

Now to calculate Tor^R_0 we need to consider the 0th homology of our tensored chain, which is of course $F_0 \otimes_R M / \mathrm{Im}(f_1 \otimes \mathrm{id})$. By the above isomorphism we know that this gives us $\mathrm{Tor}^R_0(N, M) = N \otimes_R M$. We can follow the dual method for Ext to see that we have the exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \xrightarrow{a^*} \mathrm{Hom}_R(F_0, M) \xrightarrow{b^*} \mathrm{Hom}_R(F_1, M)$$

and then taking the 0'th homology from

$$\mathrm{Hom}_R(F_1, M) \xleftarrow{b^*} \mathrm{Hom}_R(F_0, M) \leftarrow 0$$

we get that $\mathrm{Ext}^0_R(N, M) = \mathrm{Ker}(b^*) = \mathrm{Im}(a^*) \cong \mathrm{Hom}_R(N, M)$.

For a more concrete example let's work over $R = \mathbb{Z}$ and calculate $\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$. Using the free resolution

$$0 \to \mathbb{Z} \xrightarrow{\times \mathrm{n}} \mathbb{Z} \to \frac{\mathbb{Z}}{n\mathbb{Z}} \to 0,$$

we need to consider the 1'st homology of the chain

$$0 \to \mathbb{Z} \otimes \frac{\mathbb{Z}}{m\mathbb{Z}} \xrightarrow{\times \mathrm{n} \otimes \mathrm{id}} \mathbb{Z} \otimes \frac{\mathbb{Z}}{m\mathbb{Z}} \to 0.$$

This gives $\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathrm{Ker}(\times \mathrm{n} \otimes \mathrm{id} : \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z})$. Using the standard isomorphism $\mathbb{Z} \otimes G \cong G$ for any group G, we see that

$$\begin{aligned}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) =& \mathrm{Ker}(\times \mathrm{n} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}) \\ =& \frac{\mathbb{Z}}{gcd(m,n)\mathbb{Z}}.\end{aligned}$$

2.3 Tensoring a Chain with a Projective Module

We now explore what happens to the homology when a chain is tensored with a projective module. This will help us later when we are working with the Hochschild-Serre spectral sequence.

Theorem 1. *For a chain complex $C_\bullet$ of left R-modules, and P a projective right R-module we have that $H_n(P \otimes_R C_\bullet) \cong P \otimes_R H_n(C_\bullet)$.*

Proof. First in the case of P a free R-module and a chain $C_\bullet$, since $P = R^n$ we have $P \otimes_R H_n(C_\bullet) \cong H_n(P \otimes_R C_\bullet)$ by using the fact that $R \otimes_R C_\bullet \cong C_\bullet$ and that homology commutes with direct sums.

Now for P a projective R-module, we know P is a direct summand of some free R-module F. As such we have inclusion and projection maps

$$P \overset{i}{\hookrightarrow} F \overset{r}{\twoheadrightarrow} P,$$

with $ri = id_P$. From its functoriality, we can induce $(ri)_*$ on homology such that $(ri)_* = r_* i_* = id_*$. We can also map $\phi : P \otimes H_n(C_\bullet) \to H_n(P \otimes C_\bullet)$ by sending $p \otimes [x] \mapsto [p \otimes x]$, giving the following commutative diagram

$$\begin{tikzcd}
P \otimes H_n(C_\bullet) \arrow[r, "i_* \otimes id"] \arrow[d, "\phi"'] & F \otimes H_n(C_\bullet) \arrow[r, "r_* \otimes id"] \arrow[d, "\cong"'] & P \otimes H_n(C_\bullet) \arrow[d, "\phi"'] \\
H_n(P \otimes C_\bullet) \arrow[r, "(i \otimes id)_*"] & H_n(F \otimes C_\bullet) \arrow[r, "(r \otimes id)_*"] & H_n(P \otimes C_\bullet),
\end{tikzcd}$$

where the top and bottom rows both compose to the identity, and the middle map is an isomorphism from the free module case. The commutativity allows us to construct a two sided inverse for ϕ, and so ϕ is an isomorphism, giving $P \otimes H_n(C_\bullet) \cong H_n(P \otimes C_\bullet)$. ☐

Example: Suppose we want $\mathrm{Tor}_n^{\mathbb{Z}/12\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$. We know $\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, so $\mathbb{Z}/3\mathbb{Z}$ is a projective $\mathbb{Z}/12\mathbb{Z}$-module. We have the resolution

$$\cdots \xrightarrow{x2} \mathbb{Z}/12\mathbb{Z} \xrightarrow{x6} \mathbb{Z}/12\mathbb{Z} \xrightarrow{x2} \mathbb{Z}/12\mathbb{Z} \twoheadrightarrow \mathbb{Z}/6\mathbb{Z} \to 0$$

which we will call $C_\bullet \to \mathbb{Z}/6\mathbb{Z}$. This has $H_0(C_\bullet) = \mathbb{Z}/6\mathbb{Z}$ and $H_n(C_\bullet) = 1$ otherwise. So we can easily calculate our Tor values as homology commutes with our projective module. So

$$\begin{aligned}
\mathrm{Tor}_n^{\mathbb{Z}/12\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) &= H_n\left(C_\bullet \otimes_{\mathbb{Z}/12\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}\right) \\
&= H_n(C_\bullet) \otimes_{\mathbb{Z}/12\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \\
&= \begin{cases} \mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}/12\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} & n = 0, \\ \mathbb{Z}/3\mathbb{Z} & n > 0. \end{cases}
\end{aligned}$$

2.4 Simplicial Objects

[WEI, §8.1] Another concept we will need is that of a simplicial object. We can generalise the idea of a simplex to any category by selecting certain objects and then defining face and degeneracy maps that behave as we would expect them to in a geometric setting. For a category $\mathbf{C}$, then for each $n \geq 0$ we select objects $A_n \in \mathbf{C}$ along with 'face maps' $d_n : A_n \to A_{n-1}$ and 'degeneracy maps' $\sigma_n : A_n \to A_{n+1}$ which interact in the following way:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \qquad \text{for } i < j \\
\sigma_i \sigma_j &= \sigma_j \sigma_{i+1} \qquad \text{for } i \leq j,
\end{aligned}$$

and:

$$\begin{aligned}
d_i \sigma_j &= \sigma_{j-1} d_i \qquad \text{for } i < j \\
d_i \sigma_j &= id \qquad \text{for } i = j, j+1 \\
d_i \sigma_j &= \sigma_j d_{i-1} \qquad \text{for } i > j+1.
\end{aligned}$$

It should be noted that you can also think of a simplicial object as a functor $F : \Delta \to \mathbf{C}$ where the category Δ has objects the sets $\{0, \ldots, n\}$ and specially defined morphisms which then factor in a way analogous to our face and degeneracy maps. For more detail on this see [WEI, §8].

2.5 The Dold Kan Correspondence

[GOE, §III.2 prop. 2.2] We also have a way to obtain a simplicial object from a projective resolution, which we will make use of in our study of Schlichting's paper. The 'Dold Kan Correspondence' is a natural correspondence between two categories, which we can use to get a simplicial abelian group (i.e. a simplicial object from the category of abelian groups) from a chain complex.

Given a chain complex $(P_n)_{n\geq 0}$ of positively indexed abelian groups we can define a simplicial abelian group T by

$$T_n := \bigoplus_{[n]\to[k] \text{ surjective}} P_k$$

where by $[n]$ we mean the ordered set $\{0, 1, 2, \ldots, n\}$. We then get our maps from $T_n \to T_m$ by starting with an order preserving map $\theta : [m] \to [n]$ and for each indexing element $\sigma : [n] \to [k]$ of T_n, we split the composition $\sigma \circ \theta$ into a composition of an epimorphism and a monomorphism (this is called epi-monic factorisation, see [WEI lemma 8.1.2] for more detail) $[m] \overset{q}{\twoheadrightarrow} [s] \overset{i}{\hookrightarrow} [k]$ which now lets us map $P_k \xrightarrow{i_*} P_s \hookrightarrow T_m$. We do this for each indexing element σ to end up with a map $\theta^* : T_n \to T_m$. (For more detail on how to construct i_* see [GOE, §III.2], just after the proof of Theorem 2.1. For this project we will not need such details).

3 Group Homology

We now take some time to study in depth some particular constructions and results of group homology theory which we will need for Schlichting's paper. Note the common shorthand that when we have a group G, the term G-module means a $\mathbb{Z}G$-module.

3.1 Defining Group Homology

[BRO, §II.2] First we explore the different ways in which group homology can be defined. Each definition has its own merits depending on what calculations you are wanting to perform.

Definition 3. *For G and M a (left) G-module, we define the nth homology group of G as*

$$H_n(G, M) := \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M)$$

Another way to define $H_n(G, M)$ is to use the co-invariants of M.

Definition 4. *For M a G-module, the co-invariants of M is the object M_G which is defined to be M quotiented out by $\{gm - m : g \in G, m \in M\}$.*

There is also a natural isomorphism $M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$ by $[m] \mapsto 1 \otimes m$, with $\mathbb{Z}$ as a trivial right G-module and M still as a left G-module. For F a chain complex of G-modules it is then natural to denote F_G as the chain complex with $(F_G)_i = (F_i)_G$. In particular for $F_\bullet \to \mathbb{Z}$ a projective G-resolution we can then say $H_n(G, M) = H_n((F \otimes M)_G) = H_n(F \otimes_G M)$ where the first tensor product is over $\mathbb{Z}$.

Example: As a simple first example $H_0(G, M) = \mathrm{Tor}_0^{\mathbb{Z}G}(\mathbb{Z}, M) = \mathbb{Z} \otimes_{\mathbb{Z}G} M = M_G$.

Note that we can make similar definitions using Ext instead of Tor and arrive at group cohomology, though for this project it will not be needed.

3.2 A Standard Projective Resolution

[BRO, §I.5] For working with group homology it would be good to have a standard $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$ that we can work with. For a group G, we can define a projective $\mathbb{Z}G$-resolution $EG_\bullet \to \mathbb{Z}$, with $EG_n := \mathbb{Z}[G^{n+1}]$. These are $\mathbb{Z}G$-modules using the G-action $g \cdot (g_0, \dots, g_n) := (g_0 g, \dots, g_n g)$. We then have natural maps

$$\begin{aligned} d_n :\ & EG_n \to EG_{n-1} \\ (g_0, \dots, g_n)\ & \mapsto (g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n) \end{aligned}$$

which we can then assemble into a differential

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i : EG_n \to EG_{n-1}.$$

We see $\partial^2 = 0$ with a standard computation analogous to [HAT, Lemma 2.1], and we use the augmentation map $\epsilon : EG_0 \to \mathbb{Z}$, to get that $EG_\bullet$ is certainly a chain complex. To see the projective resolution $EG_\bullet \to \mathbb{Z}$ is exact it remains to check $\operatorname{Ker} \partial^n \subseteq \operatorname{Im} \partial^{n+1}$ and $\operatorname{Ker} \epsilon \subseteq \operatorname{Im} \partial^1$, which I have calculated in more detail than [BRO, §I.5]. Firstly note that if $(g_0, \dots, g_n) \in \operatorname{Ker} \partial^n$ then we can see that

$$(g_0, \dots, g_n) = \begin{cases} (g_0, g_0, g_2, g_2, \dots, g_{n-1}, g_{n-1}) & n \text{ odd} \\ (g_0, 0, g_2, 0, \dots, 0, g_n) & n \text{ even} \end{cases}$$

and so we then get

$$(g_0, \dots, g_n) = \begin{cases} \partial^{n+1}(0, g_0, 0, g_2, \dots, 0, g_{n-1}, 0) & n \text{ odd} \\ \partial^{n+1}(0, g_0, 0, g_2, \dots, 0, g_{n-1}) & n \text{ even} \end{cases}$$

so have $\operatorname{Ker} \partial^n \subseteq \operatorname{Im} \partial n + 1$ for all n, since the differentials extend linearly over sums of these basis elements. Now finally, $\operatorname{Ker} \epsilon =< (h) - (g) : h, g \in G >$ and $\partial^1(g, h) = (h) - (g)$ so $\operatorname{Ker} \epsilon = \operatorname{Im} \partial^1$. It's trivial to see $\mathbb{Z} = \operatorname{Im} \epsilon$, and so we have an exact sequence. Finally note the EG_n are projective $\mathbb{Z}G$-modules, as $\mathbb{Z}[G^{i+1}] \cong \oplus_{i+1} \mathbb{Z}[G]$ so it is a free $\mathbb{Z}G$-module. Hence we have shown $EG_\bullet \to \mathbb{Z}$ is a projective resolution of $\mathbb{Z}G$-modules.

This resolution is functorial as for a group homomorphism $f : G \to G'$ we can get a chain map $Ef : EG_\bullet \to EG'_\bullet$ by extending f over the tuples

$$\begin{aligned} Ef_n :& EG_n \to EG'_n \\ Ef_n(g_0, \dots, g_n) :=& (f(g_0), \dots, f(g_n)). \end{aligned}$$

This will be useful when we want to explicitly calculate what an induced map on homology does, as it can be unwieldy to work with a general projective resolution.

[KNU, Appendix A] I'd also like to mention an alternative resolution which we will denote $F_\bullet \to \mathbb{Z}$, where F_n is the free $\mathbb{Z}G$-module generated by tuples of G of the form $[g_1|g_2|\cdots|g_n]$. Again we assemble our differentials as $\partial^n := \sum(-1)^i d_i$ where

$$d_i[g_1|\cdots|g_n] = \begin{cases} g_1[g_2|\cdots|g_n] & i=0 \\ [g_1|\cdots|g_{i-1}|g_{i+1}|\cdots|g_n] & 0<i<n \\ [g_1|\cdots|g_{n-1}] & i=n. \end{cases}$$

It should be noted there are other natural ways to get resolutions, such as resolutions that repeat at a certain period, though in the end projective resolutions of a module are unique up to homotopy, so we can pick which is easiest to calculate with.

3.3 The Functoriality of Group Homology

Functoriality is a very important aspect of any homology theory, and as such we now examine the functoriality of group homology and give more detail to the construction given in [BRO, §III.8].

Consider two pairs (G, M) and (H, L) with G, H groups, M a G-module, and L an H-module. Now given a pair of maps (α, f) with $\alpha : G \to H$, $f : M \to L$ such that $f(gm) = \alpha(g)f(m)$ for $g \in G$, $m \in M$ our goal is to induce a map $H_*(G, M) \to H_*(H, L)$. In order to do this I've combined the statements and proofs of [BRO, Lemma I.7.4], [BRO, Lemma I.7.3] and [BRO, §I.7 (7.1)] to give the following simple lemma.

Lemma 1. *For $P_\bullet \to \mathbb{Z}$ and $Q_\bullet \to \mathbb{Z}$ two $\mathbb{Z}G$-projective resolutions of $\mathbb{Z}$ with differentials $\{\partial\}$ and $\{\partial'\}$ respectively, we can extend the identity map $id : \mathbb{Z} \to \mathbb{Z}$ to a chain map $\phi : P_\bullet \to Q_\bullet$.*

Proof. First set $\phi_0 = id$ and $\phi_k = 0$ for $k < 0$ to give a chain map in degree 0. We proceed by induction. Suppose that for all $i \leq n$ our ϕ_i is defined and satisfies $\partial_i'\phi_i = \phi_{i-1}\partial_i$. That gives the following commutative diagram

$$\begin{array}{ccccc} P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n & \xrightarrow{\partial_n} & P_{n-1} \\ & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} \\ Q_{n+1} & \xrightarrow{\partial_{n+1}'} & Q_n & \xrightarrow{\partial_n'} & Q_{n-1} \end{array}$$

where we then want to find a $\phi_{n+1} : P_{n+1} \to Q_{n+1}$. By the commutativity of the diagram we get that $\partial_n'\phi_n\partial_{n+1} = \phi_{n-1}\partial_n\partial_{n+1} = 0$ and hence we have that $\operatorname{Im}\phi_n\partial_{n+1} \subseteq \operatorname{Ker}\partial_n' = \operatorname{Im}\partial_{n+1}'$. This allows us to exploit the projectivity of P_{n+1} since we have

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dotted, "\exists\phi_{n+1}"'] \arrow[d, "\phi_n\partial_{n+1}"] & \\ Q_{n+1} \arrow[r, two heads, "\partial'_{n+1}"'] & \operatorname{Im}\partial'_{n+1} \arrow[r] & 0 \end{tikzcd}$$

where projectivity gives us existence of ϕ_{n+1} such that $\partial'_{n+1}\phi_{n+1} = \phi_n\partial_{n+1}$ and so we have found our extension to the chain map. So by induction we have the requisite $\phi : P_\bullet \to Q_\bullet$ extending the identity map. $\square$

To make use of this lemma, let $P_\bullet$ be a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$ and $Q_\bullet$ be a projective $\mathbb{Z}H$-resolution of $\mathbb{Z}$, then consider $Q_\bullet$ as a complex of G-modules by restriction of scalars along α. Our lemma now allows us to extend the identity map on $\mathbb{Z}$ to get a chain map $\phi : P_\bullet \to Q_\bullet$ where $\phi(gp) = \alpha(g)\phi(p)$ for $g \in G$ and $p \in P_\bullet$, due to our restriction of scalars along α.

Our ϕ can now be used to get $\phi \otimes f : P_\bullet \otimes_G M \to Q_\bullet \otimes_H L$, extended linearly over elementary tensors. I've calculated to following to show this too is a chain map. Take some $a \otimes b \in P_{n+1} \otimes_G M$, and note that ϕ being a chain map gives $\partial'\phi = \phi\partial$. We can now calculate

$$\begin{aligned} (\phi_n \otimes f) \circ (\partial_{n+1} \otimes id_M)(a \otimes b) = \; & (\phi_n \otimes f)(\partial_{n+1} a \otimes b) \\ = \; & \phi_n\partial_{n+1}(a) \otimes f(b) \\ = \; & \partial'_{n+1}\phi_{n+1}(a) \otimes id_L(f(b)) \\ = \; & (\partial'_{n+1} \otimes id_L) \circ (\phi_{n+1} \otimes f)(a \otimes b) \end{aligned}$$

to see that $\phi \otimes f$ commutes with the chains differentials and so will induce a well defined map on homology, as required. We denote the induced map $(\alpha, f)_*$ or just α_* in cases where its clear what the map f is.

Example : Consider a group and a normal subgroup $H \trianglelefteq G$ and M a G-module, and fix an element $g \in G$. We can define a pair of maps (α, f) where $\alpha : H \to H$ is conjugation by g, and $f : M \to M$ is left multiplication by g. Note that for $h \in H$ and $m \in M$ we have $f(hm) = ghm = ghg^{-1}gm = \alpha(h)f(m)$ so our pair of maps can induce $(\alpha, f)_* : H_*(H, M) \to H_*(H, M)$.

In particular, given our projective resolution $F_\bullet \to \mathbb{Z}$, the identity map on $\mathbb{Z}$ extends to a chain map $\phi : F_\bullet \to F_\bullet$, defined by $\phi_n(x) := gx$ for $x \in F_n$. Since for $a \in G$, $\phi_n(ax) = gax = gag^{-1}gx = \alpha(a)\phi_n(x)$, we see it's compatible with α. So our map on homology is induced by $\phi \otimes f$ which acts as $(\phi_n \otimes f)(x \otimes m) = gx \otimes gm$.

It's useful to use this induced map to define an action of G/H on $H_n(H, M)$. For $gH \in G/H$ we can then act on $[x] \in H_n(H, M)$ by using the (α, f) pair, so $gH \cdot [x] := (\alpha, f)_*([x])$

[BRO, Corollary III.8.2]. This will be a well defined as long as when restricted to H the action is trivial.

[BRO, Prop. III.8.1] To see this lets fix an $h \in H \subseteq G$ which yields an (α, f) as above. Taking a $\mathbb{Z}H$-resolution $F_\bullet \to \mathbb{Z}$ and $[x \otimes m] \in H_n(H, M)$ we see that $x \otimes m \in F_n \otimes_H M$, and our extended chain map will act as $x \otimes m \mapsto hx \otimes hm$. However since $F_n \otimes_H M = (F_n \otimes M)_H$ which is quotienting out by this exact diagonal action, so we see that the induced chain map is the identity.

To show this quotient action is well defined, take $gH = kH$ then $\exists h \in H$ such that $g = kh$ and

$$\begin{aligned}(gH) \cdot [x \otimes m] &= [gx \otimes gm] \\ &= [khx \otimes khm] \\ &= [kx \otimes km] \\ &= (kH) \cdot [x \otimes m],\end{aligned}$$

so our action is well defined. This action will be very useful later when we work with the Hochschild-Serre spectral sequence.

3.4 the Pontryagin Map

We will now study a specific isomorphism that is needed to understand a proof [SCH, Prop. 2.4] later. This requires quite a bit of technology and multiple new concepts which I will introduce along the way.

[Bro, §V.2 and §V.5] First note that for a group G and commutative ring k (with trivial G-action for simplification) we can collect together the $H_n(G, k)$ into

$$H_*(G, k) = \bigoplus H_n(G, k).$$

We can then add a product called the Pontryagin product, to turn $H_*(G, k)$ into an anti-commutative graded k-algebra. We obtain the product by composing two maps

$$H_p(G, M) \otimes H_q(G, M) \to H_{p+q}(G \times G, M \times M) \to H_{p+q}(G, M)$$

where for $[f_p \otimes m_1] \in H_p(G, M)$ and $[f_q \otimes m_2] \in H_q(G, M)$ we map $(f_p \otimes m) \otimes (f_q \otimes m_2) \mapsto (f_p \otimes f_q) \otimes (m_1 \otimes m_2)$ and then induce on the pair of maps (ϕ, f) where $\phi : G \times G \to G$ is the product in G and $f : M \times M \to M$ is the product in M.

Next we need a new algebraic object called the exterior algebra.

Definition 5. *Let V be a k-module. The k-algebra $\bigwedge^*(V) = \bigoplus_n \bigwedge^n(V)$, is the quotient of the tensor algebra $T^*(V) = k \oplus V \oplus (V \otimes_k V) \oplus \cdots$ by the relation $v \otimes v$. We denote elements of $\bigwedge^n(V)$ as sums of elements of the form $v_1 \wedge \cdots \wedge v_n$.*

Example: For $\mathbb{C}^3$ as a $\mathbb{C}$ vector space with standard basis $\{e_1, e_2, e_3\}$, we get $\bigwedge^*(\mathbb{C}) = \mathbb{C} \oplus \bigwedge^1(\mathbb{C}^3) \oplus \bigwedge^2(\mathbb{C}^3) \oplus \bigwedge^3(\mathbb{C}^3)$ a $\mathbb{C}$-algebra. Where $\bigwedge^1(\mathbb{C}^3)$ has basis $\{e_1, e_2, e_3\}$, $\bigwedge^2(\mathbb{C}^3)$ has basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ and $\bigwedge^3(\mathbb{C}^3)$ has basis $\{e_1 \wedge e_2 \wedge e_3\}$ and any higher $\bigwedge^n(\mathbb{C}^3)$ vanish due to the relation $v \otimes v$ we have quotiented out by and the fact that $\mathbb{C}^3$ is only 3 dimensional. If viewed as an $\mathbb{R}$ vector space we would then have exterior powers up to 6.

[BRO, V.6.1] The exterior algebra has the useful universal property that 'If A^* is a strictly anti-commutative graded k-algebra then any k-module map $V \to A^1$ extends uniquely to a k-algebra map $\bigwedge^*(V) \to A^*$'.

If we could relate group homology to exterior algebras, we would be able to use our understanding of exterior algebras to help us understand group homology. For an abelian group G we know $G \otimes k \cong H_1(G) \otimes k \cong H_1(G, k)$ so we have an isomorphism $G \otimes k \to H_1(G, k)$ which extends by the universal property, to a k-algebra map $\psi : \bigwedge^*(G \otimes k) \to H_*(G, k)$ which we call the **'Pontryagin map'**. We then have the following proposition.
Proposition 1 (BRO, V.6.4)**.** *The Pontryagin map is an isomorphism if every prime p s.t. G has p-torsion is invertible in k.*

Example: $\mathbb{Q}$ is a flat $\mathbb{Z}$-module so $H_*(\mathbb{Q}) \cong \bigwedge^*(\mathbb{Q})$, as a module over a PID is flat iff it is torsion free [ROT, Corollary 3.51].

We will now build up to proving this proposition, starting with the 'Kunneth theorem' which helps find the homology of a product of groups.
Theorem 2 (BRO, Corollary V.5.8)**.** *For G, G' abelian groups and k a principal ideal domain then there is a split exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(G,k) \otimes_k H_q(G',k) \to H_n(G \times G', k) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^k(H_p(G,k), H_q(G',k)) \to 0.$$

We will later derive this exact sequence once we've developed spectral sequences. As the sequence is split, we get the isomorphism

$$H_n(G \times G', k) \cong \left(\bigoplus_{p+q=n} H_p(G,k) \otimes_k H_q(G',k) \right) \bigoplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}_1^k(H_p(G,k), H_q(G',k)) \right).$$

Example: Lets compute some of the homology of the Klein four group $K_4 \cong C_2 \times C_2$. Once we work out the homology of $C_2 = \{1, \sigma\}$ we will have all the information we need. To work out $H_n(C_2, \mathbb{Z})$ we have the standard $\mathbb{Z}C_2$-projective resolution of $\mathbb{Z}$ using alternating multiplication by $\sigma - 1$ and $1 + \sigma$

$$\cdots \xrightarrow{\sigma-1} \mathbb{Z}C_2 \xrightarrow{1+\sigma} \mathbb{Z}C_2 \xrightarrow{\sigma-1} \mathbb{Z}C_2 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

with ϵ the group ring augmentation map. Now by our definition of group homology we are wanting to calculate $H_n(C_2, \mathbb{Z}) := \mathrm{Tor}_n^{\mathbb{Z}C_2}(\mathbb{Z}, \mathbb{Z})$, and so we apply $- \otimes_{\mathbb{Z}C_2} \mathbb{Z}$ functor to our projective resolution of $\mathbb{Z}$. However we know for any G-module M that $\mathbb{Z}G \otimes_G M \cong M$, so we are left with

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

This gives us that

$$H_n(C_2, \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 1, 3, 5, \ldots \\ 0 & n = 2, 4, 6, \ldots \\ \mathbb{Z} & n = 0. \end{cases}$$

We can repeat this calculation more generally, using C_n, and M a C_n-module. Knowing that $C_n = \langle \sigma \rangle$ for some $\sigma \in C_n$, we can construct a free $\mathbb{Z}C_n$-resolution of $\mathbb{Z}$

$$\cdots \xrightarrow{\sigma - 1} \mathbb{Z}C_n \xrightarrow{\phi} \mathbb{Z}C_n \xrightarrow{\sigma - 1} \mathbb{Z}C_n \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\phi = 1 + \sigma + \cdots + \sigma^{n-1}$. Then since we are calculating $\mathrm{Tor}_m^{\mathbb{Z}C_n}(\mathbb{Z}, M)$ we apply $- \otimes_{\mathbb{Z}C_n} M$, and by the same isomorphism as before we end up with

$$\cdots \xrightarrow{0} M \xrightarrow{\times n} M \xrightarrow{0} M \to 0,$$

which gives our homology as

$$H_m(C_n, M) = \begin{cases} M/nM & m = 1, 3, 5, \ldots \\ n\text{-torsion of M} & m = 2, 4, 6, \ldots \\ M & m = 0. \end{cases}$$

We can now return to our calculation of the homology of K_4, using as shorthand notation $H_n := H_n(C_2, \mathbb{Z})$, we can make use of the Kunneth theorem to see that

$$\begin{aligned} H_1(K_4, \mathbb{Z}) =& ((H_0 \otimes_{\mathbb{Z}} H_1) \oplus (H_1 \otimes_{\mathbb{Z}} H_0)) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_0, H_0) \\ =& (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus 0 \\ =& \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \end{aligned}$$

or that

$$\begin{aligned} H_2(K_4, \mathbb{Z}) =& ((H_0 \otimes_{\mathbb{Z}} H_2) \oplus (H_2 \otimes_{\mathbb{Z}} H_0) \oplus (H_1 \otimes_{\mathbb{Z}} H_1)) \oplus (\mathrm{Tor}_1^{\mathbb{Z}}(H_0, H_1) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_1, H_0)) \\ =& (\mathbb{Z} \otimes_{\mathbb{Z}} 0) \oplus (0 \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\ =& \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

The last piece we need before we can prove Prop. 1, is how to define a 'limit of groups'. We start by defining a suitable indexing set to take our limit over.

Definition 6. *For a set X with a preorder $\leq$ we call $(X, \leq)$ a directed set if for every*

$a, b \in X$ *we can find some* $c \in X$ *such that* $a \leq c$ *and* $b \leq c$.

Example : So $(\mathbb{Z}, \leq)$ is clearly a directed set, however so is its power set $\mathcal{P}(\mathbb{Z})$ using $\subseteq$ as the preorder.

Now for such a directed set X, if we are given a set of groups $\{G_\alpha\}_{\alpha \in X}$ along with homomorphisms $f_{\alpha\beta} : G_\alpha \to G_\beta$ for every $\alpha \leq \beta$ in X, where these homomorphisms satisfy that for $\alpha \leq \beta \leq \gamma$ in X then $f_{\beta\gamma} f_{\alpha\beta} = f_{\alpha\gamma}$ and $f_{\alpha\alpha} = id$. With this set-up we can now define $\varinjlim G_\alpha$.

Definition 7. *The group* $\varinjlim G_\alpha$ *is defined to be the quotient of the disjoint union* $\sqcup G_\alpha$ *by the relation that for* $g \in G_\alpha$ *and* $h \in G_\beta$ *then* $g \sim h$ *if and only if* $\exists \gamma$ *such that* $f_{\alpha\gamma}(g) = f_{\beta\gamma}(h)$. *With addition in* $\varinjlim G_\alpha$ *using these maps* $[g] + [h] := [f_{\alpha\gamma}(g) + f_{\beta\gamma}(h)]$.

Example: Consider the directed set $(\mathbb{Z}, \leq)$ mentioned above, indexing $\{G_i\}_{i \in \mathbb{Z}} := \{\mathbb{Z}^i\}_{i \in \mathbb{Z}}$ with the natural inclusions $f_{ij} : \mathbb{Z}^i \to \mathbb{Z}^j$, for $i \leq j$. Then we see that elements of our $\varinjlim G_\alpha$ are equivalence classes of tuples of integers. So for instance we have $[(1, 0, 1, 1)] \sim [(1, 0, 1, 1, 0, 0)]$ and we can add, by lifting to a shared power of $\mathbb{Z}$, $[(1, 0, 1, 1)] + [(0, 0, 2, 2, 2, 0, 1)] = [(1, 0, 3, 3, 2, 0, 1)]$. We see that in this case $\varinjlim G_\alpha$ is effectively the group of integer sequences with finitely many non-zero terms.

With this technology we can now prove Prop. 1, that the Pontryagin map is an isomorphism when every prime p such that G has p-torsion, is invertible in k.

Proof. [BRO, V.6.4] We will break this proof down into three cases. First is the case that G is cyclic. Then we see $\bigwedge^p(G \otimes k) = 0$ for all $p > 1$, and with our p-torsion hypothesis we get $H_p(G, k) = 0$ for $p > 1$ by our computation of $H_m(C_n, M)$ we performed in the Kunneth formula example, and hence our map is an isomorphism.

Now for the case that G is finitely generated, we induct. By the fundamental theorem of finitely generated abelian groups we can induct on the number of cyclic factors of G. For the base case that $G = G_1 \times G_2$ we can create the following commutative diagram

$$\begin{array}{ccc} \bigwedge^*(G_1 \otimes k) \otimes \bigwedge^*(G_2 \otimes k) & \xrightarrow[\cong]{\varphi} & \bigwedge^*(G \otimes k) \\ {\scriptstyle \psi(G_1) \otimes \psi(G_2)} \big\downarrow & & \big\downarrow {\scriptstyle \psi(G)} \\ H_*(G_1, k) \otimes_k H_*(G_2, k) & \xrightarrow{\mu} & H_*(G, k) \end{array}$$

Where μ is from the splitting of the Kunneth short exact sequence (§3.4, Thm 2), the ψ are the extended k-algebra maps we are concerned with and the top map is from the isomorphism $\bigwedge^*(A \oplus B) \cong \bigwedge^*(A) \otimes \bigwedge^*(B)$ which in our case gives $\bigwedge^*(G_1 \otimes k) \otimes \bigwedge^*(G_2 \otimes k) \cong \bigwedge^*((G_1 \otimes k) \oplus (G_2 \otimes k)) \cong \bigwedge^*((G_1 \times G_2) \otimes k) \cong \bigwedge^*(G \otimes k)$.

To see the diagram commutes we rely on the naturality of ψ. To expand upon Brown's justificaiton: in particular ψ is a natural transformation between the two functors $F := \bigwedge^*(\ \cdot \otimes k)$ and $K := H_*(\ \cdot\ , k)$, so for any group homomorphism $f : A \to B$ we get

$\psi(B) \circ F(f) = K(f) \circ \psi(A)$. We also have inclusion maps $i_j : \bigwedge^*(G_j \otimes k) \to \bigwedge^*(G \otimes k)$ for $j = 1, 2$ and also $f_{j*} : H_*(G_j, k) \to H_*(G, k)$ for $j = 1, 2$ induced by the natural inclusions of groups G_1, G_2 into G. So taking an element $a \otimes b \in \bigwedge^*(G_1 \otimes k) \otimes \bigwedge^*(G_2 \otimes k)$, the two paths of the diagram are

$$\begin{aligned}(\psi(G) \circ \varphi)(a \otimes b) &= \psi(G)(i_1 a \cdot i_2 b) \\ &= \psi(G)(i_1 a) \cdot \psi(G)(i_2 b)\end{aligned}$$

and

$$\begin{aligned}(\mu \circ \psi(G_1) \otimes \psi(G_2))(a \otimes b) &= \mu(\psi(G_1)(a) \otimes \psi(G_2)(b)) \\ &= f_{1*}\psi(G_1)(a) \cdot f_{2*}\psi(G_2)(b).\end{aligned}$$

Applying our naturality to the i_j and f_{j*} maps we see that we have $f_{j*} \circ \psi(G_j) = \psi(G) \circ i_j$ and hence the two routes of our diagram are equal.

Now by the cyclic case and our theorems hypothesis, we get that $\psi(G_1)$ and $\psi(G_2)$ are isomorphisms, and $G_1 \otimes k$ and $G_2 \otimes k$ are k-free, so the Tor terms in the Kunneth short exact sequence vanish giving μ is an isomorphism and so we get that ψ is an isomorphism. The rest of the induction follows easily from this.

Now the final case is when $G = \varinjlim G_\alpha$, for G_α finitely generated. Similar to above we have a commutative diagram

$$\begin{tikzcd} \varinjlim \bigwedge^*(G_\alpha \otimes k) \arrow[r, "\varphi"] \arrow[d, "\varinjlim \varphi(G_\alpha)"'] & \bigwedge^*(G \otimes k) \arrow[d, "\psi(G)"] \\ \varinjlim H_*(G_\alpha, k) \arrow[r, "\mu"] & H_*(G, k) \end{tikzcd}$$

Where the bottom map comes from $G_\alpha \hookrightarrow G$ inducing $\mu : \varinjlim H_*(G_\alpha, k) \xrightarrow{\cong} H_*(G, k)$ [BRO, ex V.5.3]. The top map, φ, is and isomorphism induced by the inclusions $G_\alpha \otimes k \hookrightarrow G \otimes k$ [Brown, V.6.3]. Finally again, as above, by naturality of ψ the diagram commutes and so we are done. $\square$

4 Spectral Sequences

Now we have a solid grasp of the basic tools of group homology we move onto defining and working with spectral sequences. The motivating example of why we need spectral sequences is in wanting to compute the homology of the total complex of the tensor product of two chain complexes, which we shall see later in this section.

4.1 Definition of a Spectral Sequence

[ROT, §10.2] We start first with the definition of a spectral sequence. The next few sections then define exact couples and show how to actually obtain our spectral sequences from an exact couple, and what it means for a spectral sequence to converge. We start with the following definitions.

Definition 8 (ROT, pg610, pg619)**.** *A differential bigraded module is an ordered pair (M, d) where $M = (M_{p,q})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ is a bigraded module and d is a bigraded map such that $d^2 = 0$.*

These objects then make up our spectral sequence.

Definition 9 (ROT, pg622)**.** *A spectral sequence is a sequence of differential bigraded modules*

$$(E^r, d^r)_{r\in\mathbb{N}}$$

such that each consecutive module is related to the pervious one by

$$E^r \cong H(E^{r-1}, d^{r-1}).$$

We call the E^r the pages of the spectral sequence and d^r the differentials.

In more concrete terms, if d^{r-1} has bidegree (a, b) then we get

$$E^r_{p,q} \cong \frac{\operatorname{Ker} d^{r-1}_{p,q}}{\operatorname{Im} d^{r-1}_{p-a,q-b}}.$$

N.B. This is the homological form of a spectral sequence, there is also a cohomological form.

4.2 The Convergence of a Spectral Sequence

[ROT, §10.3] We now need to define the limiting behaviour as $r \to \infty$. Intuitively the limiting page, should be the point where moving to the next page no longer has any effects. So naively we could make the following definition

Definition 10 (Convergence version 1)**.** *If there exists an $n \in \mathbb{N}$ such that for all $r \geq n$ we have that $E^r = E^n$ then we call this page E^∞.*

However there is a more robust way that we have to define E^∞. First notice that since each $E^r = H(E^{r-1}, d^{r-1}) = \operatorname{Ker} d^{r-1}/\operatorname{Im} d^{r-1}$ we can denote the r'th page as a quotient of objects, $E^r = Z^r/B^r$ using the letters Z and B from the topological notion of homology being 'cycles modulo boundaries'. Due to the fact that we are taking repeated quotients as we progress from one page to the next, we see that

$$B^2 \subseteq \cdots \subseteq B^r \subseteq Z^r \subseteq \cdots \subseteq Z^2 \subseteq E^1.$$

These objects can then be collected together to define

$$Z^\infty := \bigcap_r Z^r$$
$$B^\infty := \bigcup_r B^r$$

and these let us arrive at our definition for E^∞ [ROT, pg624].

Definition 11 (Convergence version 2)**.** *We define the E^∞ limiting term of our spectral sequence to be the quotient*

$$E^\infty := Z^\infty / B^\infty$$

Using this E^∞ we can now define what it means for our spectral sequence to converge.

Definition 12 (ROT, pg626)**.** *We say our spectral sequence converges to a graded object H_* if there is a bounded filtration Φ of H_* such that:*

$$E^\infty_{p,q} \cong \frac{\Phi^p H_{p+q}}{\Phi^{p-1} H_{p+q}},$$

where by a bounded filtration of H we mean that for each H_n, we can find integers $m < u$ (possibly dependant on n) such that $0 = \Phi^m H_n \subseteq \Phi^{m+1} H_n \subseteq \cdots \subseteq \Phi^u H_n = H_n$. We denote this as $E^2_{p,q} \Rightarrow H_n$ or $E^2_{p,q} \underset{p}{\Rightarrow} H_n$ when we want to emphasise the fact that we are using p to control the filtration.

[ROT, lemma 10.13] Its worth noting that if the pages stabilise after a point then our naive definition of convergence is the same as our definition that uses cycles and boundaries. To see this first note that if there is an r such that the pages stabilise then we have $E^r = E^{r+1}$ meaning that $E^r = Z^{r+1}/B^{r+1}$ however by definition we also have $B^{r+1} \subseteq Z^{r+1} \subseteq E^r$, so the only way for that equality to hold is if $B^{r+1} = 0$ but E^r is itself a quotient Z^r/B^r and so to be 0 in this quotient gives us $B^{r+1} = B^r$. On top of this $E^r = Z^r/B^r = E^{r+1} = Z^{r+1}/B^{r+1} = Z^{r+1}/B^r$ and so that forces $Z^r = Z^{r+1}$ too.

Now as $E^r = E^{r+1} = E^{r+2} = \cdots$ we have equality for those cycles and boundaries too and hence for every $s \geq r$ due to the subquotient nature of our definitions we get

$$B^2 \subseteq \cdots \subseteq B^r = \cdots = B^s \subseteq Z^s = \cdots = Z^r \subseteq \cdots Z^2$$

and so by definition

$$B^\infty = \bigcup_n B^n = \bigcup_{n \geq r} B^n = B^r,$$

and

$$Z^\infty := \bigcap_n Z^n = \bigcap_{n \geq r} Z^r = Z^r,$$

and hence

$$E^\infty = B^\infty/Z^\infty = B^r/Z^r = E^r = E^{r+1} = \cdots$$

which is the point where our sequence stabilises. In the most common cases our spectral sequence will be zero outside of a single quadrant, which along with the changing bidegree of differentials will cause the pages to stabilise after a finite number of iterations.

4.3 Exact Couples

[ROT, §10.2] Exact couples are the main object we can use to obtain a spectral sequence.

Definition 13. *An exact couple is a commutative triangle of bigraded modules, with bigraded module homomorphisms as below*

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

which is exact at each point, so $\operatorname{Ker}\alpha = \operatorname{Im}\gamma$*,* $\operatorname{Ker}\beta = \operatorname{Im}\alpha$ *and* $\operatorname{Ker}\gamma = \operatorname{Im}\alpha$*. We denote our exact couple as the tuple* $(D, E, \alpha, \beta, \gamma)$*.*

From an initial exact couple we can obtain a sequence of exact couples, which we will call derived couples, and these will give us the pages for a spectral sequence.

To get our derived couple we first define a differential $d^1 = \beta\gamma : E \to E$ and denote the homology with respect to d^1 as $E^2 := H(E, d^1)$. We also define $D^2 := \operatorname{Im}(\alpha)$. We then have a map $\alpha^2 := \alpha|_{D^2} : D^2 \to D^2$ the restriction of the α from the original exact couple.

We also have a map $\beta^2 : D^2 \to E^2$ which is defined as follows. For every $x \in D^2$ we can find a $y \in D$ such that $x = \alpha(y)$ by definition of $\operatorname{Im}(\alpha)$. We then define $\beta^2(x) := [\beta(y)]$. To see this is independent of the choice of y, suppose that we can find $y_1 \neq y_2$ such that $x = \alpha(y_1) = \alpha(y_2)$. This tells us that $\alpha(y_1 - y_2) = 0$ so $y_1 - y_2 \in \operatorname{Ker}(\alpha) = \operatorname{Im}(\gamma)$, so $\exists z \in E$ with $y_1 - y_2 = \gamma(z)$. Applying β we see this means $\beta(y_1) = \beta(y_2) + \beta \circ \gamma(z)$, which tells us $[\beta(y_1)] = [\beta(y_2)] \in E^2 = H(E, \beta \circ \gamma)$ so our map is well defined.

Finally we have a map $\gamma^2 : E^2 \to D^2$ that is naturally induced by γ on the homology of E. That is, for some $[z] \in H(E, d^1)$, so $z \in \operatorname{Ker}(d^1) = \operatorname{Ker}(\beta \circ \gamma)$, we have $\gamma^2([z]) := \gamma(z)$. To see γ^2 has the correct range note that $z \in \operatorname{Ker}(\beta \circ \gamma)$ means $\gamma(z) \in \operatorname{Ker}(\beta) = \operatorname{Im}(\alpha)$ by exactness, so $\gamma^2(x) = \gamma(z) \in D^2$ as required. To see it is well defined, suppose we had two elements such that $[x] = [y]$ in E^2. Then that would mean $x - y \in \operatorname{Im}(d^1) = \operatorname{Im}(\beta \circ \gamma)$, so we can find a $z \in E$ such that $x - y = \beta(\gamma(z))$. But then $\gamma^2([x]) - \gamma^2([y]) = \gamma(x - y) = \gamma(\beta(\gamma(z))) = 0$ since by exactness $\gamma \circ \beta = 0$. So the map induced by γ on homology is well defined.

Which all together gives us

$$\begin{tikzcd}
D^2 = \mathrm{Im}(\alpha) \arrow[rr, "\alpha^2"] & & D^2 = \mathrm{Im}(\alpha) \arrow[dl, "\beta^2"] \\
& E^2 = H(E, d^1) \arrow[ul, "\gamma^2"] &
\end{tikzcd}$$

All that is left to do is show exactness at each point to confirm it's an exact couple.

The above and following calculations are my own, though analogous ones can be found in [ROT, Prop. 10.9]. To see that we have exactness we need to show three things: (i) $\mathrm{Ker}\,\alpha^2 = \mathrm{Im}\,\gamma^2$, (ii) $\mathrm{Ker}\,\beta^2 = \mathrm{Im}\,\alpha^2$, and finally (iii) $\mathrm{Ker}\,\gamma^2 = \mathrm{Im}\,\beta^2$, which we will now proceed to do.

(i) First to see $\mathrm{Ker}\,\alpha^2 \subset \mathrm{Im}\,\gamma^2$

$$\begin{aligned}
x \in \mathrm{Ker}\,\alpha^2 \Rightarrow\ & \alpha(x) = 0 \\
\Rightarrow\ & x \in \mathrm{Ker}\,\alpha = \mathrm{Im}\,\gamma \\
\Rightarrow\ & \exists y \in E \text{ s.t. } \gamma(y) = x \\
\Rightarrow\ & x = \gamma^2([y]) \text{ as } y \in \mathrm{Ker}\,\beta\gamma. \text{ [Check?]}
\end{aligned}$$

Then to see $\mathrm{Im}\,\gamma^2 \subset \mathrm{Ker}\,\alpha^2$

$$\begin{aligned}
x \in \mathrm{Im}\,\gamma^2 \Rightarrow\ & \exists [y] \in H(E, d^2) \text{ s.t. } \gamma^2([y]) = \gamma(y) = x \\
\Rightarrow\ & x \in \mathrm{Im}\,\gamma = \mathrm{Ker}\,\alpha \\
\Rightarrow\ & x \in \mathrm{Ker}\,\alpha^2 = \mathrm{Ker}\,\alpha|_{D^2} \subset \mathrm{Ker}\,\alpha.
\end{aligned}$$

(ii) First to see $\mathrm{Ker}\,\beta^2 \subset \mathrm{Im}\,\alpha^2$

$$\begin{aligned}
x \in \mathrm{Ker}\,\beta^2 \Rightarrow\ & \beta^2(x) = 0 \\
\Rightarrow\ & \exists y \in D \text{ s.t. } \alpha(y) = x \text{ so } \beta^2(x) := \beta(y) = 0 \\
\Rightarrow\ & y \in \mathrm{Ker}\,\beta = \mathrm{Im}\,\alpha \\
\Rightarrow\ & y \in D^2 := \mathrm{Im}\,\alpha \text{ and } \alpha(y) = x \text{ so } x \in \mathrm{Im}\,\alpha^2.
\end{aligned}$$

Then to see $\mathrm{Im}\,\alpha^2 \subset \mathrm{Ker}\,\beta^2$

$$\begin{aligned}
x \in \mathrm{Im}\,\alpha^2 \Rightarrow\ & \exists y \in \mathrm{Im}\,\alpha \text{ s.t. } \alpha^2(y) := \alpha(y) = x \\
\Rightarrow\ & x \in \mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta \\
\Rightarrow\ & \beta^2(x) = \beta(y) = 0.
\end{aligned}$$

(iii) First to see $\operatorname{Ker}\gamma^2 \subset \operatorname{Im}\beta^2$

$$\begin{aligned}
[x] \in \operatorname{Ker}\gamma^2 \Rightarrow\ & x \in \operatorname{Ker}\gamma = \operatorname{Im}\beta \\
\Rightarrow\ & \exists y \in D \text{ s.t. } \beta(y) = x \\
\Rightarrow\ & \beta^2(\alpha(y)) := [\beta(y)] = [x].
\end{aligned}$$

Then to see $\operatorname{Im}\beta^2 \subset \operatorname{Ker}\gamma^2$

$$\begin{aligned}
[x] \in \operatorname{Im}\beta^2 \Rightarrow\ & \exists y \in D^2 \text{ s.t. } \beta^2(y) = [x] \\
\Rightarrow\ & \exists z \in D \text{ s.t. } \alpha(z) = y \text{ and } \beta^2(y) = [\beta(z)] = [x] \\
\Rightarrow & \gamma^2([x]) = \gamma^2([\beta(z)]) = \gamma(\beta(z)) = 0 \text{ as } \beta(z) \in \operatorname{Im}\beta = \operatorname{Ker}\gamma.
\end{aligned}$$

Together this gives us exactness of our derived couple, so as stated we have obtained a new exact couple $(D^2, E^2, \alpha^2, \beta^2, \gamma^2)$ from our original exact couple.

From this we define the n'th derived couple, $(D^n, E^n, \alpha^n, \beta^n, \gamma^n)$, inductively as the derived couple of $(D^{n-1}, E^{n-1}, \alpha^{n-1}, \beta^{n-1}, \gamma^{n-1})$. As each E^n is defined as the homology of E^{n-1}, we get a spectral sequence by collecting the (E^n, d^n) from each couple. The task is now to find some useful exact couples to work with!

4.4 The Spectral Sequence from the Filtration of a Chain Complex

This section will give a general method that we can use to obtain useful exact couples to work with. Given a chain complex C a filtration of C is a sequence of sub-complexes F^pC such that

$$\cdots \subseteq F^{p-1}C \subseteq F^pC \subseteq F^{p+1}C \subseteq \cdots \subseteq C.$$

where we also require that $d(F^pC^n) \subseteq F^pC^{n-1}$.

There is a natural way to get an exact couple from a filtration of a chain complex, using the short exact sequence of chain complexes with the usual inclusion and quotient maps

$$0 \to F^{p-1}C \to F^pC \to F^pC/F^{p-1}C \to 0.$$

We know a short exact sequence of chain complexes gives a long exact sequence in homology

$$\cdots \to H_nF^{p-1}C \xrightarrow{\alpha} H_nF^pC \xrightarrow{\beta} H_nF^pC/F^{p-1}C \xrightarrow{\gamma} H_{n-1}F^{p-1}C \to \cdots$$

with α induced by the inclusion map, β induced by the quotient map and γ the connecting homomorphism. We can now define bigraded modules

$$\begin{aligned}
D_{p,q} :=& H_{p+q}(F^pC) \\
E_{p,q} :=& H_{p+q}(F^pC/F^{p-1}C)
\end{aligned}$$

which give an exact couple $(D, E, \alpha, \beta, \gamma)$. We have exactness since our maps come from the long exact sequence. We can see α has bidegree $(1, -1)$, β has bidegree $(0, 0)$ and γ has bidegree $(-1, 0)$. From this we can take the derived couples $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$, with α^r of bidegree $(1, -1)$, β^r of bidegree $(1 - r, r - 1)$, γ^r of bidegree $(-1, 0)$, and at each stage the differential d^r has bidegree $(-r, r - 1)$.

The (E^r, d^r) from the derived couples give us a spectral sequence. We now show that this spectral sequence converges to the homology of C, i.e. to the graded object $H_n(C)$. By definition, that means we need to find a bounded filtration Φ^p of H_n such that $E^\infty_{p,q} \cong \Phi^p H_{p+q}(C)/\Phi^{p-1} H_{p+q}(C)$. To do this, I've combined and added details to several parts of Rotman's text including [ROT, Thm. 10.16], [ROT, Corollary 10.10] and [ROT, Thm. 10.14].

Lets first start by noticing that given our filtration F^pC of C, then for each p we have a natural inclusion map

$$i_p : F^pC \hookrightarrow C$$

which can induce a map on homology for each n

$$(i_p)_* : H_n(F^pC) \to H_n(C)$$

and this gives us a filtration of $H_n(C)$ which we can define for each p and n as

$$\Phi^p H_n(C) := \mathrm{Im}\,(i_p)_*$$

which will be bounded if our F^p was a bounded filtration. So assuming F^p was bounded, we now have a bounded filtration of $H_*(C)$ which we need to relate to E^∞. To get our hands on E^∞ we will work with our filtration through the D^r terms of the derived couple.

Recall that from $(D, E, \alpha, \beta, \gamma)$, the r'th derived couple consists of the following

$$\begin{tikzcd} D^r \arrow[rr, "\alpha^r"] & & D^r \arrow[dl, "\beta^r"] \\ & E^r \arrow[ul, "\gamma^r"] & \end{tikzcd}$$

with exactness at each point. In particular for each p we get an exact sequence

$$D^r_{p-2+r,q-r+2} \xrightarrow{\alpha^r} D^r_{p-1+r,q-r+1} \xrightarrow{\beta^r} E^r_{p,q} \xrightarrow{\gamma^r} D^r_{p-1,q}$$

since α^r has bidegree $(1, -1)$, β^r has bidegree $(1 - r, r - 1)$ and γ^r has bidegree $(-1, 0)$.

Also note that following our definition for the D^r we can see

$$\begin{aligned} D^r_{p,q} &= \alpha^{r-1} D^{r-1}_{p-1,q+1} \\ &= \alpha^{r-1}\alpha^{r-2} D^{r-2}_{p-2,q+2} \\ &= \dots \\ &= \alpha^{r-1} \cdots \alpha^2 \alpha D_{p-(r-1),q+(r-1)} \end{aligned}$$

and from our filtration we defined $D_{p,q} := H_{p+q}(F^pC)$ and α the map induced by the inclusion $j^{p-1} : F^{p-1}C \hookrightarrow F^pC$. So our chain of α^r is the same as chaining all these inclusions together and then inducing the map, so we see

$$\begin{aligned} D^r_{p,q} &= \operatorname{Im}(j^{p-1}_* j^{p-2}_* \cdots j^{p-r+1}_*) \\ &= \operatorname{Im}(j^{p-1} j^{p-2} \cdots j^{p-r+1})_* \subseteq H_{p+q}(F^pC). \end{aligned}$$

where that composition maps $H_{p+q}(F^{p-r+1}C) \to H_{p+q}(F^pC)$.

This tells us that $D^r_{p-1+r,q-r+1} \subseteq H_{p+q}(F^{p-1+r}C)$ and for large enough r we will have $F^{p-1+r}C = C$ since its a bounded filtration. Because of this, the composition of inclusions is actually equal to our previously seen inclusion map $i^p : F^pC \to C$.

We can now apply this information to our exact sequence. Firstly we can say for large r we have $D^r_{p+r-1,\#} = \operatorname{Im}(i^p)_* =: \Phi^p H_{p+q}(C)$ (where # denotes the second coordinate as it doesn't matter what it exactly is here) and similarly $D^r_{p-2+r,q-r+2} = \Phi^{p-1}H_{p+q}(C)$. Secondly we see that the final term in the exact sequence

$$D^r_{p-1,q} = \operatorname{Im}(H_{p-1+q}(F^{p-r}C) \to H_{p-1+q}(F^{p-1}C))$$

will actually be 0 for large enough r since the boundedness of the filtration means eventually $F^{p-r}C = 0$.

When r is large enough that both of these occur, we get pieces of the form

$$0 \to \Phi^{p-1}H_{p+q}(C) \to \Phi^p H_{p+q}(C) \to E^r_{p,q} \to 0,$$

and hence have that

$$E^\infty_{p,q} = E^r_{p,q} \cong \frac{\Phi^p H_{p+q}(C)}{\Phi^{p-1}H_{p+q}(C)},$$

which by definition that gives us that

$$E^2_{p,q} \underset{p}{\Rightarrow} H_{p+q}(C).$$

Now that we have gone through the above technicalities, given a chain complex and a bounded filtration of that complex we can now apply the following process

1. define an exact couple using the filtration,
2. take the derived couples of this exact couple,
3. use these derived couples to obtain the pages and differentials of a spectral sequence which converges to the homology of the chain complex.

4.5 Maps of Exact Couples

[MAS, §5] It's worth quickly noting how we can define a map of exact couples.

Definition 14. *Given two exact couples $(D, E, \alpha, \beta, \gamma)$ and $(F, G, \alpha', \beta', \gamma')$, we define a map of exact couples as a pair of maps*

$$(f, g) : (D, E, \alpha, \beta, \gamma) \to (F, G, \alpha', \beta', \gamma')$$

such that $f : D \to F$ and $g : E \to G$, where these maps 'commute' with the differentials in the exact couples, in theh sense that

$$\begin{aligned} f \circ \alpha &= \alpha' \circ f \\ g \circ \beta &= \beta' \circ f \\ f \circ \gamma &= \gamma' \circ g. \end{aligned}$$

These relations mean that when we take derived couples, the differentials $d := \beta \circ \gamma$ and $d' := \beta' \circ \gamma'$ will then commute with g too, since

$$\begin{aligned} g \circ d &= g \circ \beta \circ \gamma \\ &= \beta' \circ f \circ \gamma \\ &= \beta' \circ \gamma' \circ g \\ &= d' \circ g \end{aligned}$$

and so its easy to take our pair (f, g) and induce maps on the derived couples, as this commutativity means g induces a well defined map on homology, and we can restrict f to $\operatorname{Im} \alpha$ to end up with a map between the derived couples.

A map between derived couples then gives a map of spectral sequences, where by map of spectral sequences we mean the following.

Definition 15. *Suppose (E^r, d^r) and (F^r, g^r) are two spectral sequences then a map $f : E \to F$ of spectral sequences is a collection of maps between each page $f^r : E^r \to F^r$ that commutes with the page maps $f \circ d^r = g^r \circ f$ for each r, and such that f^{r+1} is the map induced on homology by f^r.*

4.6 Example: Spectral Sequence Collapsing to an Axis

There is a special case in which the convergence of a spectral sequence tells us more explicitly about the object it is converging to. The following are my own calculations and are a good way to get used to how convergence works. Suppose we have a spectral sequence (E^n, d^n), converging to a graded object H, and that the E^∞ page 'collapses to the x-axis', so in other words

$$E^\infty_{p,q} = \begin{cases} A_p & q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We will now explore what this specific case tells us about H. Firstly, we know that for each n we have a bounded filtration

$$0 = \Phi^m H_n \subseteq \Phi^{m+1} H_n \subseteq \cdots \subseteq \Phi^{u-1} H_n \subseteq \Phi^u H_n = H_n,$$

and also that

$$E^\infty_{n,0} = \frac{\Phi^n H_n}{\Phi^{n-1} H_n} = A_n.$$

We can get more information about H_n with this filtration by exploring the other terms of $E^\infty_{p,q}$ where $p + q = n$. Looking 'above and below' it we find for each $k \geq 1$ that

$$E^\infty_{n+k,-k} = \frac{\Phi^{n+k} H_n}{\Phi^{n+k-1} H_n} = 0,$$

$$E^\infty_{n-k,k} = \frac{\Phi^{n-k} H_n}{\Phi^{n-(k-1)} H_n} = 0.$$

Which we can now chain together and make use of the boundedness property to see that

$$0 = \Phi^m H_n = \cdots = \Phi^{n-1} H_n$$

and

$$\Phi^n H_n = \Phi^{n+1} H_n = \cdots = \Phi^u H_n = H_n.$$

These together tell us that

$$E^\infty_{n,0} = \frac{\Phi^n H_n}{\Phi^{n-1} H_n} = \frac{H_n}{0} = H_n = A_n$$

for all n. So we explicitly know that our H is made up of the non-zero terms of the E^∞ page. Similar results can be deduced for the spectral sequence with E^∞ made up of any single column or row with all the other terms zero.

4.7 The Spectral Sequence of a Double Complex

We will now show how two particular filtrations of a double complex give rise to useful spectral sequences. For a double complex $M_{p,q}$ with vertical and horizontal differentials d^v and d^h respectively, satisfying $(d^h)^2 = 0$, $(d^v)^2 = 0$ and $d^h d^v + d^v d^h = 0$ we can create a chain complex called the 'total complex'.

Definition 16 (Total complex of a double complex)**.** *Given a double complex M, as above, we define the total complex of M as the complex*

$$\mathrm{Tot}(M)_n := \bigoplus_{p+q=n} M_{p,q},$$

with differentials

$$d_n : \mathrm{Tot}(M)_n \to \mathrm{Tot}(M)_{n-1}$$
$$d_n = d^v + d^h.$$

Example: One common occurrence of the total complex is when we have two chain complexes C and D of R-modules, for some ring R. It is usual to define their tensor product as

$$(C \otimes D)_n := \bigoplus_{p+q=n} C_p \otimes_R D_q.$$

However, equivalently we can define a double complex $M_{p,q} := C_p \otimes_R D_q$ and then see $C \otimes D = Tot(M)$. The homology of such a tensor product is intimately related to finding the homology of products of spaces and products of groups which is where we start to see how spectral sequences can come in useful.

The total complex comes equipped with two bounded filtrations which we now define using the notation of Rotman [ROT, §10.2].

Definition 17. *The first filtration of the total complex* $\mathrm{Tot}(M)$ *is the subcomplex*

$$({}^I F^p \, \mathrm{Tot}(M))_n := \bigoplus_{i \leq p} M_{i,n-i}.$$

The second filtration of the total complex $\mathrm{Tot}(M)$ *is the subcomplex*

$$({}^{II} F^p \, \mathrm{Tot}(M))_n := \bigoplus_{j \leq p} M_{n-j,j}.$$

We know each such filtration gives an exact couple (§4.4) and hence yields a spectral sequence. Associated to the first filtration we get a spectral sequence with

$$\begin{aligned} {}^{I}E^1_{p,q} &= H_{p+q}\left(\frac{{}^{I}F^p\,\mathrm{Tot}(M)}{{}^{I}F^{p-1}\,\mathrm{Tot}(M)}\right) \\ &= H_{p+q}(M_{p,\bullet}[p]) \\ &= H_q(M_{p,\bullet}), \end{aligned}$$

where $M_{p,\bullet}[p]$ is the p'th column with indexes shifted by p and where this page comes with differentials induced by the horizontal differentials of M. We can now chase this back one page if we want to see that

$${}^{I}E^0_{p,q} = M_{p,q}$$

using d^0 as the vertical differentials of M.

From the second filtration we get a spectral sequence with

$$\begin{aligned} {}^{II}E^1_{p,q} &= H_{p+q}({}^{II}F^pM/{}^{II}F^{p-1}M) \\ &= H_{p+q}\left(\frac{\bigoplus_{j\le p} M_{(p+q)-j,j}}{\bigoplus_{j\le p-1} M_{(p+q)-j,j}}\right) \\ &= H_{p+q}(M_{\bullet,p}[p]) \\ &= H_q(M_{\bullet,p}), \end{aligned}$$

and so similarly we can see that this came from 0'th page

$${}^{I}E^0_{p,q} = M_{q,p}$$

except this time the d^0 differentials are from the horizontal differentials of M, and then d^1 is induced by the vertical differentials of M on homology.

So the difference between the spectral sequences ${}^{I}E$ and ${}^{II}E$ is the order we take homology using the maps from M and we know they converge to $H_*(Tot(M))$. In our spectral sequence notation we can denote this as

$$\begin{aligned} {}^{I}E^2 &= H^hH^v(M) \Rightarrow H_*(\mathrm{Tot}(M)) \\ {}^{II}E^2 &= H^vH^h(M) \Rightarrow H_*(\mathrm{Tot}(M)) \end{aligned}$$

where H^v and H^h are shorthand denoting if we took homology via the vertical, d^v, or horizontal, d^h, differentials of our double complex.

[LOD, Appendix D] Its worth performing a quick analysis of the d^2 differential. We know d^0 is either the vertical or horizontal differential, for the purpose of this calculation suppose we are using the first filtration, so $d^0 = d^v$ the vertical differential and d^1 is induced on homology from d^h.

Then d^2 with bidegree $(-2,1)$ takes a little more work to understand, but we can describe it explicitly in much the same way as one would the connecting homomorphism. Pick some $[x] \in E^2_{p,q} = \operatorname{Ker} d^1_{p,q} / \operatorname{Im} d^1_{p,q}$, so we can represent $[x]$ by some $x \in E^1_{p,q}$ such that $d^1(x) = 0$. However $E^1 = \operatorname{Ker} d^0 / \operatorname{Im} d^0$, so we have that x is itself a class that we can represent by some $\tilde{x} \in \operatorname{Ker} d^0_{p,q}$ an element of the double complex.

Now note that

$$\begin{aligned} d^1(x) = 0 \Rightarrow\ & x \in \operatorname{Im}(d^0) \\ \Rightarrow\ & \exists y \in M_{p-1,q+1} \text{ s.t. } d^h(\tilde{x}) = d^v(y) \end{aligned}$$

and as $d^h d^v + d^v d^h = 0$ and $d^h d^h = 0$ we can then see

$$d^v d^h(y) = -d^h d^v(y) = -d^h d^h(\tilde{x}) = 0$$

and hence $d^h(y) \in \operatorname{Ker} d^v$ and so can define a class $[d^h(y)] \in E^1_{p-2,q+1}$. So lets consider what happens when we apply d^1 to this element.

$$d^1([d^h(y)]) := [d^h d^h(y)] = 0$$

so this element is also in $\operatorname{Ker} d^1$ and hence considering its class we get an element in $E^2_{p-2,q+1}$. Not only this but as d^2 has bidegree $(-2,1)$ this is really the only element that it makes sense to send our original $[x]$ to under d^2.

4.8 Example: Two Non-Zero Columns

We will now see another calculation example, which is my own work solving [WEI, Exercise 5.2.1]. Given a double complex M where our associated spectral sequence has E^2 all zero other than in the columns $p = 0$ and $p = 1$, we will work to find a useful short exact sequence. To start, we know

$$E^2_{p,q} \Rightarrow H_{p+q}(\operatorname{Tot}(M))$$

and that d^2 has bidegree $(-2,1)$ so taking homology wont change anything, and hence $E^2 = E^\infty$. So for any n we see that

$$E^\infty_{1,n-1} = \frac{\Phi^1 H_n}{\Phi^0 H_n} = E^2_{1,n-1},$$

and

$$E^\infty_{0,n} = \frac{\Phi^0 H_n}{\Phi^{-1} H_n} = E^2_{0,n}.$$

We can then also note that for $k \geq 1$ we have

$$E^\infty_{-k,n+k} = \frac{\Phi^{-k} H_n}{\Phi^{-k-1} H_n} = E^2_{-k,n+k} = 0.$$

So as before since the filtration is bounded it will eventually descend to zero and so we get

$$0 = \cdots = \Phi^{-2}H_n = \Phi^{-1}H_n$$

and hence we now know $\Phi^0 H_n = E^2_{0,}$. Similarly we see again for $k \geq 1$ that

$$E^\infty_{k,n-k} = \frac{\Phi^k H_n}{\Phi^{k-1}H_n} = E^2_{k,n-k} = 0$$

and by boundedness of the filtration we arrive at

$$\Phi^1 H_n = \Phi^2 H_n = \cdots = H_n(\mathrm{Tot}(M)).$$

Putting the two together we can see

$$E^\infty_{1,n-1} = \frac{\Phi^1 H_n}{\Phi^0 H_n} = \frac{H_n(\mathrm{Tot}(M))}{E^2_{0,n}} = E^2_{1,n-1},$$

and hence for each n there is a short exact sequence

$$0 \to E^2_{0,n} \to H_n(\mathrm{Tot}(M)) \to E^2_{1,n-1} \to 0.$$

We can apply this to easily derive what happens if spectral sequence collapses to the y-axis. In this case the third term of our short exact sequence is zero so we get $E^2_{0,n} \cong H_n(\mathrm{Tot}(M))$.

4.9 Example: Balancing Tor

Our spectral sequence tools also allow us to neatly solve a natural question that could have arisen when we first defined the Tor functor. This can be seen in [ROT, Cor. 10.23] which I've added detail to. For N a left R-module and M a right R-module we can compute $\mathrm{Tor}_R(M,N)$ in two ways [ROT, Thm. 7.5]. We could take a flat resolution $M_\bullet \to M$ then compute the homology of $M_\bullet \otimes N$. Alternatively we could take a flat resolution $N_\bullet \to N$ and compute the homology of $M \otimes N_\bullet$. We can prove that these two methods are equal by exploiting the two filtrations of the double complex and the fact that the spectral sequences they yield both converge to the same object.

We start by defining the double complex $C_{p,q} := M_p \otimes_R N_q$. The spectral sequence from the first filtration of this double complex (§4.7) gives

$${}^I E^0_{p,q} := C_{p,q} = M_p \otimes_R N_q.$$

Computing each vertical homology we see this leads to

$$\begin{aligned} {}^{I}E^1_{p,q} &= \frac{\mathrm{Ker}(M_p \otimes N_q \to M_p \otimes N_{q-1})}{\mathrm{Im}(M_p \otimes N_{q+1} \to M_p \otimes N_q)} \\ &= \begin{cases} M_p \otimes N & q = 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since the resolution of N is an exact sequence. Taking the homology again, this time horizontally we end up with

$${}^{I}E^2_{p,q} = \begin{cases} \mathrm{Tor}^R_p(M_\bullet, N) & q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

As all other terms are zero and the differentials are now pointing out of the horizontal line, we see $E^\infty = E^2$. Following the same steps with the second filtration we arrive at

$${}^{II}E^2_{p,q} = \begin{cases} \mathrm{Tor}^R_p(M, N_\bullet) & q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is again the E^∞ page. So both sequences collapse to an axis, so the non-zero terms are the object we are converging to. We also know that these two spectral sequences converge to the same object and so we get that $\mathrm{Tor}^R(M_\bullet, N) = \mathrm{Tor}^R(M, N_\bullet)$.

4.10 A Spectral Sequence Proof of the Kunneth Formula

[ROT, Thm. 10.90] Now we are used to making basic calculations with spectral sequences, we can use them to derive the short exact sequence seen in the Kunneth formula (§3.4, Thm. 2). We take as given that for two positively indexed chain complexes A and C or R-modules we have a spectral sequence

$$E^2_{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}^R_p(H_s(A), H_t(C)) \Rightarrow H_n(A \otimes_R C).$$

We will assume flatness of the cycles and boundaries in the homology of A.

We start with the standard exact sequence of cycles Z_s and boundaries B_s,

$$0 \to B_s \to Z_s \to H_s(A) \to 0.$$

This gives us a long exact sequence in Tor that we saw in §2.1

$$\cdots \to \mathrm{Tor}^R_p(B_s, H_t(C)) \to \mathrm{Tor}^R_p(Z_s, H_t(C)) \to \mathrm{Tor}^R_p(H_s(A), H_t(C)) \to \mathrm{Tor}^R_{p-1}(B_s, H_t(C)) \to \cdots.$$

The flatness of cycles and boundaries then causes the vanishing of the terms

$$\operatorname{Tor}_p^R(B_s, H_t(C)) = \operatorname{Tor}_p^R(Z_s, H_t(C)) = 0 \quad \text{for } p \geq 1,$$

so in our long exact sequence we end up being left with

$$\cdots \to 0 \to 0 \to \operatorname{Tor}_p^R(H_s(A), H_t(C)) \to 0 \to \cdots$$

when $p \geq 1$, which of course tells us that unless $p = 0, 1$ we have

$$\operatorname{Tor}_p^R(H_s(A), H_t(C)) = 0.$$

So we end up with the second page of our spectral sequence having non-zero terms in only two columns, which is a scenario we have already analysed in more generality. As we calculated in §4.8, for every n we have a short exact sequence

$$0 \to E^2_{0,n} \to H_n(A \otimes_R C) \to E^2_{1,n-1} \to 0.$$

In particular this gives us

$$0 \to \bigoplus_{s+t=n} \operatorname{Tor}_0^R(H_s(A), H_t(C)) \to H_n(A \otimes_R C) \to \bigoplus_{s+t=n-1} \operatorname{Tor}_1^R(H_s(A), H_t(C)) \to 0$$

and we have calculated Tor_0^R in §2.2 so know that this becomes

$$0 \to \bigoplus_{s+t=n} H_s(A) \otimes_R H_t(C) \to H_n(A \otimes_R C) \to \bigoplus_{s+t=n-1} \operatorname{Tor}_1^R(H_s(A), H_t(C)) \to 0. \qquad (1)$$

In particular if we take $F_\bullet \to \mathbb{Z}$, $Q_\bullet \to \mathbb{Z}$ as $\mathbb{Z}G$ and $\mathbb{Z}G'$ resolutions respectively, and note that $F_\bullet \otimes Q_\bullet \to \mathbb{Z}$ is a resolution for $G \times G'$. Then define our chains as $A := (F_\bullet \otimes k)_G$ and $C := (Q_\bullet \otimes k)_{G'}$, to see equation (1) above gives the short exact sequence we wanted for the Kunneth formula (§3.4, Thm. 2).

Note: Brown proves the kunneth formula in a very different manner, via some explicitly constructed maps, for this version see [BRO, V.5.8]. I chose this method to showcase the technology we have been building up. Also note that the spectral sequence we used here comes from that of the double complex but involves defining a Cartan-Eilenberg projective resolution first [ROT, §10.5], which would take us too far from the topic at hand.

4.11 The Hochschild-Serre Spectral Sequence

We will now describe the main spectral sequence we need for our work on Schlichting's paper. This spectral sequence is an example of the more general Grothendiek spectral sequence, relating to the composition of two functors [ROT, §10.6]. We start with the notion of a group extension.

Definition 18. *A group extension is a short exact sequence of groups*

$$1 \to N \to G \to K \to 1.$$

and we say G is an extension of K by N.

Example : We can have an extension of the cyclic group of order two

$$1 \to D_3 \to D_6 \to C_2 \to 1,$$

or we could have an extension of the integers modulo p, by localising at at the prime ideal $\langle p \rangle$ to give us

$$0 \to \langle p \rangle \mathbb{Z}_{\langle p \rangle} \to \mathbb{Z}_{\langle p \rangle} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Now we can see how a group extension can provide us with a spectral sequence.

Theorem 3. *Given such a group extension $0 \to N \to G \to K$ and a G-module M, there is a spectral sequence called the Hochschild-Serre spectral sequence, given by*

$$E^2_{p,q} := H_p(K, H_q(N, M)) \Rightarrow H_{p+q}(G, M).$$

[BRO, §III.8.2] Note in order to define $H_p(K, H_q(N, M))$ we need $H_q(N, M)$ to be a K-module. We saw earlier (§3.3) that G acts via conjugation on $H_n(N, M)$, and as N acts trivially on $H_n(N, M)$, we have a G/N action on $H_n(N, M)$ and by our group extension $G/N \cong K$. This makes $H_q(N, M)$ into a K-module as required.

I have now combined multiple statements from Brown's text indlucing [BRO, §VII.5] and [BRO, §VII.6] to derive this spectral sequence in detail. We use the spectral sequence of a filtration of a double complex which we saw in §4.7. First let $F_\bullet \to \mathbb{Z}$ a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$. We are going to exploit the fact that

$$F \otimes_G M = (F \otimes M)_G = ((F \otimes M)_N)_K = (F \otimes_N M)_K$$

to compute the homology of G. We start with the chain $C_q := (F_q \otimes M)_N$, and see by definition $H_n(G, M) = H_n(F \otimes_G M) = H_n(C_K)$. So we focus on calculating $H_n(C_K)$.

Letting $L_\bullet \to \mathbb{Z}$ be a projective $\mathbb{Z}K$-resolution, we can define the double complex $L \otimes_K C$, in particular

$$\begin{aligned} (L \otimes_K C)_{p,q} = &\ L_p \otimes_K C_q \\ = &\ L_p \otimes_{\mathbb{Z}K} (F_q \otimes M)_N. \end{aligned}$$

We start by studying the first filtration spectral sequence of this double complex, so $E^0_{p,q} = (L \otimes C)_{p,q}$ which we can see as

$$
\begin{array}{c|cccc}
3 & \vdots & \vdots & \vdots & \\
 & \downarrow & \downarrow & \downarrow & \\
2 & L_0 \otimes_{\mathbb{Z}K} (F_2 \otimes M)_N & L_1 \otimes_{\mathbb{Z}K} (F_2 \otimes M)_N & L_2 \otimes_{\mathbb{Z}K} (F_2 \otimes M)_N & \cdots \\
 & \downarrow & \downarrow & \downarrow & \\
1 & L_0 \otimes_{\mathbb{Z}K} (F_1 \otimes M)_N & L_1 \otimes_{\mathbb{Z}K} (F_1 \otimes M)_N & L_2 \otimes_{\mathbb{Z}K} (F_1 \otimes M)_N & \cdots \\
 & \downarrow & \downarrow & \downarrow & \\
0 & L_0 \otimes_{\mathbb{Z}K} (F_0 \otimes M)_N & L_1 \otimes_{\mathbb{Z}K} (F_0 \otimes M)_N & L_2 \otimes_{\mathbb{Z}K} (F_0 \otimes M)_N & \cdots \\
\hline
 & 0 & 1 & 2 &
\end{array}
$$

Taking homology we get to the E^1 page. Now since the L_i are projective and so commute when we take homology (§2.3, Thm. 1), and recalling that $H_q(C) = H_q(N, M)$, we arrive at

$$
\begin{array}{c|ccccccc}
3 & \vdots & & \vdots & & \vdots & & \\
2 & L_0 \otimes_{\mathbb{Z}K} H_2(N, M) & \leftarrow & L_1 \otimes_{\mathbb{Z}K} H_2(N, M) & \leftarrow & L_2 \otimes_{\mathbb{Z}K} H_2(N, M) & \leftarrow & \cdots \\
1 & L_0 \otimes_{\mathbb{Z}K} H_1(N, M) & \leftarrow & L_1 \otimes_{\mathbb{Z}K} H_1(N, M) & \leftarrow & L_2 \otimes_{\mathbb{Z}K} H_1(N, M) & \leftarrow & \cdots \\
0 & L_0 \otimes_{\mathbb{Z}K} H_0(N, M) & \leftarrow & L_1 \otimes_{\mathbb{Z}K} H_0(N, M) & \leftarrow & L_2 \otimes_{\mathbb{Z}K} H_0(N, M) & \leftarrow & \cdots \\
\hline
 & 0 & & 1 & & 2 & &
\end{array}
$$

Taking homology once more gives the terms on the second page

$$E^2_{p,q} = H_p(K, H_q(N, M)).$$

In order to prove this converges to $H_{p+q}(G, M)$, we now examine the spectral sequence from the second filtration, $E^0_{p,q} = (L \otimes C)_{q,p}$. Taking homology we get $E^1_{p,q} = H_q(K, C_p)$. Picking F to be the standard resolution (§3.2), we can focus on studying $H_q(K, \mathbb{Z}G \otimes_N M) = \mathrm{Tor}_q^{\mathbb{Z}K}(\mathbb{Z}, \mathbb{Z}G \otimes_N M)$. As a K-module we see that $\exists M_0$ such that $\mathbb{Z}G \otimes_N M \cong \mathrm{Ind}_N^G \mathrm{Res}_N^G(M) \cong \mathbb{Z}K \otimes M_0$, with the latter isomorphism from [BRO, III.5.6]. Taking $P_\bullet \to M_0$ a free resolution of M_0, then $\mathbb{Z}K \otimes P_\bullet \to \mathbb{Z}K \otimes M_0$ can be used to calculate Tor. However $\mathbb{Z} \otimes_K \mathbb{Z}K \otimes F_\bullet \cong F_\bullet$ and so Tor vanishes for $q > 0$. So the only non-zero terms remaining are $E^1_{p,0} = H_0(K, C_p) = (C_p)_K$. This means E^2 collapses to the x-axis as

$H_n(C_K)$, and both spectral sequences converge to the same object, giving

$$E^2_{p,q} = H_p(K, H_q(N, M)) \Rightarrow H_{p+q}(C_K) \cong H_{p+q}(G, M).$$

So we have fully derived the Hochschild-Serre spectral sequence from the spectral sequences from a double complex.

4.12 Induced Map on Hochschild-Serre

In order to understand [SCH, Prop 2.4] we need some more detail on how a morphism on the Hochschild-Serre spectral sequence is induced by a map of the group extension. Given a map f that is an automorphism of the short exact sequence used in the Hostchild-Serre spectral sequence

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "f_N"'] & G \arrow[r] \arrow[d, "f_G"'] & K \arrow[r] \arrow[d, "f_K"'] & 0 \\
0 \arrow[r] & N \arrow[r] & G \arrow[r] & K \arrow[r] & 0,
\end{tikzcd}$$

we can induce a natural map on the spectral sequence itself. (The following is my own calculation) We will do this by giving a map that commutes with the differentials of the bicomplex $M_{p,q} = EK_p \otimes_K (EG_q \otimes M)_N = EK_p \otimes_K (EG_q \otimes_N M)$, where $EK_\bullet \to \mathbb{Z}$ is our standard resolution of $\mathbb{Z}K$-modules from §3.2, with differentials ∂^K, and $EG_\bullet \to \mathbb{Z}$ is our standard resolution of $\mathbb{Z}G$-modules with differentials ∂^G. Such a map will will induce a map on the total complex, and hence the filtrations of the total complex. From there maps can be induced on the exact couple from that filtration, which is the exact couple we obtain the Hochschild-Serre spectral sequence from. Note that this complex has horizontal differential $d^h = \partial^K \otimes id \otimes id$ and vertical differential $d^v = id \otimes \partial^G \otimes id$.

Our induced map $Ef_K : EK_\bullet \to EK_\bullet$ is a chain map, so $\partial^K Ef_K = Ef_K \partial^K$. We can do the same to get a chain map $Ef_G : EG_\bullet \to EG_\bullet$. Tensoring these together gives $Ef_K \otimes Ef_G \otimes id : M_{p,q} \to M_{p,q}$, and we see it commutes with the differentials of M by

$$\begin{aligned}
(Ef_K \otimes Ef_G \otimes id) \circ (d^h) =& \ (Ef_K \otimes Ef_G \otimes id) \circ (\partial^K \otimes id \otimes id) \\
=& \ (Ef_K \partial^K \otimes Ef_G \otimes id) \\
=& \ (\partial^K Ef_K \otimes Ef_G \otimes id) \\
=& \ (\partial^K \otimes id \otimes id) \circ (Ef_K \otimes Ef_G \otimes id) \\
=& (d^h) \circ (Ef_K \otimes Ef_G \otimes id)
\end{aligned}$$

and a similar calculation shows that it commutes with d^v.

This map of M can easily extend to $C := \mathrm{Tot}(M)$ and then be restricted to the filtrations F^pC. Hence we get a map of the short exact sequence of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & F^{p-1}C \arrow[r] \arrow[d, "F"'] & F^pC \arrow[r] \arrow[d, "F"'] & F^pC/F^{p-1}C \arrow[r] \arrow[d, "F"'] & 0 \\
0 \arrow[r] & F^{p-1}C \arrow[r] & F^pC \arrow[r] & F^pC/F^{p-1}C \arrow[r] & 0,
\end{tikzcd}$$

which induces a map of the long exact sequence in homology, and hence gives us a map of our exact couple, which in turn gives a map on the spectral sequence as we have already seen.

4.13 Hochschild-Serre with a Split Short Exact Sequence

There is another special case of the Hochschild-Serre spectral sequence that's worth considering. That of when the group extension splits

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & G \arrow[r, "r"] & K \arrow[l, bend left, "i"] \arrow[r] & 0,
\end{tikzcd} \tag{2}$$

where $ri = id$. In this case we will see that we have convergence at $H_i(K) = E^2_{i,0} = E^\infty_{i,0}$. In order to see this we make use of two Hochschild-Serre spectral sequences. We will call the Hochschild-Serre spectral sequence from the above group extension (eqn. 2), $E(G)$. A second group extension

$$0 \longrightarrow 0 \longrightarrow G/N \longrightarrow G/N \longrightarrow 0, \tag{3}$$

gives a second Hochschild-Serre spectral sequence, $E(G/N)$. We see on the E^2 page that $E^2_{i,j}(G/N) = H_i(G/N, H_j(0))$. These terms are trivial so we have that $E^2_{i,0} \cong E^\infty_{i,0}$. We can now use the splitting of eqn. 2, and the fact that homology is functorial to give the following commutative diagram

$$\begin{tikzcd}
E^2_{i,0}(G/N) \arrow[r, "i_*"] \arrow[d, two heads, "\cong"'] & E^2_{i,0}(G) \arrow[r, "r_*"] \arrow[d, two heads] & E^2_{i,0}(G/N) \arrow[d, two heads, "\cong"'] \\
E^\infty_{i,0}(G/N) \arrow[r, "i_*"] & E^\infty_{i,0}(G) \arrow[r, "r_*"] & E^\infty_{i,0}(G/N).
\end{tikzcd}$$

The top and bottom rows compose to the identity, and the outer two vertical maps are isomorphisms, so we get that the middle map is an isomorphism too. This is the result we wanted. We will see this result come in useful when we work with the proof technique of Schlichting's [SCH, Thm. 2.5].

5 Euler Class Paper

We can now start to work with the paper of Schlichting [SCH]. The main results of this paper concern the area of homology stability. Homology stability has been an important concept of study over the last few decades. Homology stability is concerned with when in a sequence of groups $G_1 \subseteq G_2 \subseteq \cdots$, does the homology $H_i(G_k)$ become independent of k. This information allows us to calculate the homology of groups such as $GL(R) = \varinjlim GL_n(R)$. Groups like $GL(R)$ are intimately linked to the algebraic K-theory of R.

The main result we will focus on concerns the homology stability of $SL(A)$ for a special type of ring A that we will define. The result is stated as

Theorem 4 (SCH, Thm. 3.12)**.** *Let A be a commutative ring with many units and $n \geq 2$ an integer. Then the natural homomorphism*

$$H_i(SL_{n-1}(A), \mathbb{Z}) \to H_i(SL_n(A), \mathbb{Z})$$

is an isomorphism for $n \geq i + sr(A) + 1$ and surjective for $n \geq i + sr(A)$, where $sr(A)$ is a quantity called the stable rank of A.

This theorem needs multiple technological arguments and lemmas to prove it, and it is two of these we will focus on to see how our spectral sequences can be applied. Note: we will not define $sr(A)$ here as it won't directly concern us, but [VAS] has more detail if required.

5.1 Definitions and Notation

The underlying object that allows many of these results to occur is that of a 'ring with many units'.

Definition 19. *Let A be a ring and $m \geq 0$ an integer. We call A a ring with many units if for all such m we can find a sequence of elements $(a_1, \dots, a_n) \subset Z(A)$ whose non-empty partial sums are units. We will refer to such a sequence as an $S(m)$-sequence in A.*

Example: Local ring with infinite residue fields, so in particular any field, are examples of rings with many units.

We will also use $S(m)$ to refer to the commutative ring

$$S(m) := \mathbb{Z}[X_1, \ldots, X_m][\Sigma^{-1}]$$

where

$$\Sigma := \{\sum_{j \in J} X_j : \emptyset \neq J \subset [1, m] := \{1, \ldots, m\}\}$$

which has an $S(m)$ sequence of $(X_1, \ldots, X_m)$. For an $S(m)$-sequence $(a_1, \ldots, a_m)$ in a ring A and for $\emptyset \neq J \subset [1, m]$ we denote the partial sums as $a_J := \sum_{j \in J} a_j$ and define the elements

$$s(a) := -\sum_{\emptyset \neq J \subset [1,m]} (-1)^{|J|} \langle a_J \rangle \in \mathbb{Z}[A^*].$$

and

$$s_t(a) := -\sum_{\emptyset \neq J \subset [1,m]} (-1)^{|J|} \langle (a_J)^t \rangle \in \mathbb{Z}[A^*].$$

In the particular case that $A = S(m)$ we then will use the notation $s_m := s(X_1, \ldots, X_m)$ and $s_{m,t} := s_t(X_1, \ldots, X_m)$, following Schlichting.

5.2 An Alternative Proof of Schlichting's Proposition 2.4

The result of the following proposition is of vital use in [SCH]. Here I prove it using more direct methods than the topologically focused ones used in [SCH].

Proposition 2 (SCH, Prop. 2.4)**.** *Let R be a commutative ring with $S(m)$-sequence $a = (a_1, \ldots, a_m)$. let M be an R-module. Then for all integers $t, q \geq 1$ with $tq < m$ the integral homology groups $H_q(M, \mathbb{Z})$ of M are $s_t(a)$-torsion, that is, localisation at $s_t(a)$ yields*

$$[s_t(a)^{-1}] H_q(M, \mathbb{Z}) = 0.$$

Proof. We can say WLOG that $R = S(m)$ and $(a_1, \ldots, a_m) = (X_1, \ldots, X_m)$ since R having an $S(m)$ sequence makes R an $S(m)$-algebra and hence we can me M an $S(m)$ module, so we can work with trying to localise at $s_t(X)$.

Now view M as the simplicial $S(m)$-module, $i \mapsto M$. Using the Dold Kan correspondence on a projective resolution of M we get a weak equivalence of simplicial $S(m)$-modules $P_* \to M$. The classifying space functor then yields an $S(m)^*$-equivariant weak equivalence of the simplicial sets $BP_* \to BM$ and so we have $H_n(BP_*) \cong H_n(BM)$. We can now use the double complex $C_{i,j} \mapsto \mathbb{Z}[B_i P_j]$ to help us calculate $H_n(BP_*)$ since

$$H_n(BP_*) = H_n(\mathrm{Tot}(C)).$$

In particular we can use the spectral sequence from the second filtration of this complex

$$E^0_{r,s} := \mathbb{Z}[B_s P_r]$$

with bidegree $(0, -1)$ differentials from the alternating sums of face maps from the simplicial module $i \mapsto B_i P_r$. This gives us that

$$E^1_{r,s} = H_s(BP_r) \Rightarrow H_{r+s}(\mathrm{Tot}(C_\bullet)) \cong H_{r+s}(BM).$$

As $S(m)$ is flat over $\mathbb{Z}$, and each P_r is projective, we get that eac P_r is torsion free, making the Pontryagin map

$$\Lambda^s_{\mathbb{Z}} P_r \to H_s(BP_r)$$

an isomorphism of $S(m)^*$-modules as we proved in §3.4, Prop. 1. By two technical results [SCH, Lemma 2.2] and [SCH, Corollary 2.3] which we will admit for brevity and in order to focus on the spectral sequence aspect of the proofs, we have that for $1 \leq ts < m$ these $\Lambda^s_{\mathbb{Z}} P_r$ are $s_t(X)$-torsion. We can localise the spectral sequence to get a new sequence $[s_t(X)^{-1}]E^1_{r,s} \Rightarrow [s_t(X)^{-1}]H_{r+s}(\mathrm{Tot}(C_\bullet))$. In particular the localised E^1 and E^2 pages are then

$$\begin{array}{c|cccc}
\hline
\frac{m}{s} & 0 & 0 & 0 & \cdots \\
 & \cdots & 0 & \cdots & \\
1 & 0 & 0 & 0 & \cdots \\
\hline
0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots \\
\hline
 & 0 & 1 & 2 & \cdots
\end{array}
\qquad
\begin{array}{c|cccc}
\hline
\frac{m}{s} & 0 & 0 & 0 & \cdots \\
 & \cdots & 0 & \cdots & \\
1 & 0 & 0 & 0 & \cdots \\
\hline
0 & \mathbb{Z} & 0 & 0 & \cdots \\
\hline
 & 0 & 1 & 2 & \cdots
\end{array}$$

$[s_t(X)^{-1}]E^1$ $\qquad$ $[s_t(X)^{-1}]E^2$

We get the bottom row of $[s_t(X)^{-1}]E^2$ by noticing that the horizontal differentials in $[s_t(X)^{-1}]E^1_{\bullet,0}$ are alternating sums of the identity map and so alternate between the identity and the zero map. Therefore, for $1 \leq tq < m$ the localised spectral sequence converges to 0, so in that range $[s_t(X)^{-1}]H_q(\mathrm{Tot}(C_\bullet)) = 0$ as required. $\square$

5.3 A Simplified Proof of Nesterenko and Suslin

We now reprove a result that was originally proved by Nesterenko and Suslin in [NES, Thm. 1.11], but here we present an alternate proof that adapts the techniques of Schlichting [SCH, Thm. 2.5], in which a similar result is proved for an analogue of $SL_q(A)$ instead of $GL_q(A)$. The $SL_q(A)$ result is more technical, especially when A is noncommutative, as it requires defining an analogue of the determinant map to even work out what we should mean by $SL_q(A)$. This proof can be seen as a stepping stone in understanding [SCH, Thm. 2.5] in that the general strategy is the same but less technicalities obstruct us as our group action is better behaved here.

In order to state our result, we first define the following

$$Aff^{GL}_{p,q}(A) := \begin{pmatrix} GL_q(A) & 0 \\ M_{p,q}(A) & 1_p \end{pmatrix}.$$

Theorem 5. *Let A be an $S(m)$-algebra. Let $q \geq 1$ an integer, then for all integers $p, r \geq 0$ such that $0 \leq r < m$ the inclusion*

$$\iota : GL_q(A) \to Aff^{GL}_{p,q}(A) : M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1_p \end{pmatrix}$$

induces an isomorphism of A^*-modules

$$H_r(GL_q(A)) \cong H_r(Aff^{GL}_{p,q}(A))$$

Proof. We have the short exact sequence

$$0 \to M_{p,q}(A) \to \begin{pmatrix} GL_qA & 0 \\ M_{p,q}A & 1_p \end{pmatrix} \to GL_q(A) \to 1. \tag{4}$$

From this we get the Hochschild-Serre spectral sequence associated to the above short exact sequence

$$E^2_{i,j} = H_i(GL_qA, H_j(M_{p,q}A)) \Rightarrow H_{i+j}(Aff^{GL}_{p,q}(A)).$$

We want to make use of the previous proposition to see where certain parts of our spectral sequence vanish once we localise it at the element $s_{m,-1}$. In particular as A is an $S(m)$-algebra, it has an $S(m)$ sequence $a := (a_1, \ldots, a_m)$ for $a_i \in Z(A)$, combinations of which make up $s_{m,-1}$ so we need to know how the a_i^{-1} act on the spectral sequence. In particular they act as the blocksums (where $\mathrm{diag}_q(x_1, \ldots, x_n)$ is the $q \times q$ diagonal matrix with entries

$x_i)$

$$\begin{aligned}
&\mathrm{diag}_q(a_i^{-1},\dots,a_i^{-1})\oplus I_p \in Aff_{p,q}^{GL}(A) && \text{by conjugation on } Aff_{p,q}^{GL}(A),\\
&\mathrm{diag}_q(a_i^{-1},\dots,a_i^{-1}) \in GL_q(A) && \text{by conjugation on } GL_q(A),\\
&\mathrm{diag}_q(a_i^{-1},\dots,a_i^{-1})^{-1} \in GL_q(A) && \text{by right multiplication on } M_{p,q}(A),
\end{aligned}$$

however we already know that $Aff_{p,q}^{GL}(A)$ and $GL_q(A)$ act trivially on the spectral sequence, so $s_{m,-1}$ acts trivially on the spectral sequence. Note that $\mathrm{diag}_q(a_i^{-1},\dots,a_i^{-1})^{-1} = \mathrm{diag}_q(a_i,\dots,a_i)$ so on $H_j(M_{p,q}(A))$ we have $s_{m,-1}^{-1}$ acting as $s_{m,1}^{-1}$ which is what allows us to use the previous proposition §5.2 Prop. 2. Also as the $a_i \in Z(A)$ the conjugation action is trivial on $GL_q(A)$.

Since this action of $s_{m,-1}$ is trivial on homology we see that our localised spectral sequence actually converges to the same homology as the original spectral sequence as

$$[s_{m,-1}^{-1}]E_{i,j}^2 \Rightarrow [s_{m,-1}^{-1}]H_{i+j}(Aff_{p,q}^{GL}(A)) \cong H_{i+j}(Aff_{p,q}^{GL}(A))$$

where we have the isomorphism due to the action being trivial. In particular we can see what happens on the E^2 page and also make use of the proposition we just proved (§5.2, Prop 2) and also the fact that $H_0(M_{p,q}A) = \mathbb{Z}$, to get

$$\begin{aligned}
H_i(GL_qA, H_j(M_{p,q}A)) &= [s_{m,-1}^{-1}]H_i(GL_qA, H_j(M_{p,q}A))\\
&= H_i(GL_qA, [s_{m,1}^{-1}]H_j(M_{p,q}A))\\
&= \begin{cases} H_i(GL_q(A), 0) = 0 & \text{for } 0 < j < m,\\ H_i(GL_q(A), \mathbb{Z}) & \text{for } j = 0,\end{cases}
\end{aligned}$$

as on $\mathbb{Z}$ our action is through the augmentation map and $\epsilon(s_{m,1}) = 1$ so it acts as the identity.

As our group extension (§5.3, eqn. 4) splits, we know from §4.13 that $H_i(GL_q(A),\mathbb{Z}) = E_{i,0}^2 = E_{i,0}^\infty$. Combining our calculations we see that we end up with $[s_{m,-1}^{-1}]E_{i,j}^2 = [s_{m,-1}^{-1}]E_{i,j}^\infty = 0$ for $0 < j < m$, and $[s_{m,-1}^{-1}]E_{i,0}^2 = [s_{m,-1}^{-1}]E_{i,0}^\infty = H_i(GL_q(A),\mathbb{Z})$. Taking $0 \le r < m$ we can follow through the filtrations as before to see the isomorphism we need. In particular

$$[s_{m,-1}^{-1}]E_{r,0}^\infty \cong \frac{\Phi^r H_r(Aff_{p,q}^{GL}(A))}{\Phi^{r-1}H_r(Aff_{p,q}^{GL}(A))} = H_r(GL_q(A))$$

and due to our complex being only non-zero in the first quadrant and Φ induced from the first and second filtrations of the double complex we get

$$0 = \Phi^{-1}H_r(Aff_{p,q}^{GL}(A)) = \dots = \Phi^{r-1}H_r(Aff_{p,q}^{GL}(A))$$

and

$$H_r(GL_q(A)) = \Phi^r H_r(Aff^{GL}_{p,q}(A)) = H_r(Aff^{GL}_{p,q}(A))$$

where the isomorphism is induced by the inclusion $GL_q(A) \hookrightarrow Aff^{GL}_{p,q}(A)$ by the definition (from §4.4) of our filtration Φ. $\square$

5.4 Making Use of These Results

To finish we will briefly touch on how these results are used in [SCH] to give the homology stability for $SL_q(A)$. First Schlichting makes use of the K-theoretic determinant map to define a group called $SG_q(A)$ which is analagous to $SL_q(A)$ when A is noncommutative. The $SG_q(A)$ version of §5.3 Thm. 5 gives the isomorphism $H_r(SG_q(A)) \cong [s^{-1}_{m,-t}]H_r(Aff^{SG}_{p,q}(A))$ for a certain range of r, t, m, q [SCH, Thm. 2.5]. This is then rephrased in the more general homological algebra context of total derived functors [SCH, Cor. 2.6].

A specific chain complex and spectral sequence are then constructed, and it is shown that parts of its first page are isomorphic to the homology of $SG_q(A)$ [SCH, lemma 3.3]. With some more in depth analysis of this spectral sequence Schlichting is able to conclude a homology stability result about $SG_q(A)$ [SCH, Thm. 3.7]. Some more work is then needed which involves showing that in a certain range $SG_q(A) = E_q(A)$, the group generated by elementary matrices [SCH, Lemma 3.8]. Finally taking A to be commutative, the homology stability of $SL_q(A)$ follows as a consequence of that of $SG_q(A)$ [SCH, Thm. 3.12].

6 Further Reading

There are multiple directions the reader could head in from the topics in this dissertation:

1. Further reading on spectral sequences. They have many applications not only in K-theory but in topology, differential geometry (see [BTU]) and in algebraic geometry to name a few. The section in [ROT] has many more examples and applications, as does the latter half of [BRO].

2. Further reading on K-theory. This dissertation really only begins the technical work that underpins one part of K-theory. There is a huge amount of the subject surrounding the homology stability of these groups which involves a lot more interesting and deep algebra, see [WEK].

3. Further reading on group homology. [BRO] is the standard text. It is a large and fascinating, with much more than what we could cover here.

4. Further reading on homological algebra. [ROT] and [WEI] are both dense texts which touch on category theory and more powerful and complex objects as well as

the general theory of specific objects like project, injective and flat modules to name a few.

5. Further reading of [SCH]. The paper builds using the technical results we used here to give homology stability results about $SL_n(A)$ as well as some more general results. Much of this requires reading from the above suggestions to understand.

Bibliography

[SCH] M.Schlichting, *Euler class groups and the homology of elementary and special linear groups*, Advances in Mathematics 320 (2017), 1-81.

[AWO] S.Awodey, *Category Theory, second edition*, Oxford Logic Guides 52, 2008.

[ROT] J.J.Rotman, *An Introduction to Homological Algebra, second edition*, Springer, 2009.

[WEI] C.A.Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, 2008.

[BRO] K.S.Brown, *Cohomology of Groups*, Springer Graduate Texts, 1982.

[KNU] K.P.Knudson, *Homology of Linear Groups*, Progress in Mathematics vol. 193, Birkhauser, 2000.

[GOE] P.G.Goerss and J.F.Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics vol. 174, Birkhauser, 1999.

[HAT] A,Hatcher, *Algebraic Topology*, Cambridge University Press, 2009.

[MAS] W.S.Massey, *Exact couples in algebraic toplogy parts I and II*, Annals of Mathematics, Second Series, Vol. 56, No.2 (Sep, 1952), pp. 363-396, section 5.

[LOD] J.L.Loday, *Cyclic Homology*, Springer-Verlag, 1998.

[VAS] L.N. Vaserstein, *The stable range of rings and the dimension of topological spaces*, Funktsional. Anal. i Prilozhen. 5(2) (1971) 1727.

[NES] Yu.P.Nesterenko and A.A.Suslin, *Homology of the Full Linear Group Over a Local Ring, and Milnor's K-Theory*, Mathematics of the USSR-Izvestiya vol. 34 pg 121, 1990.

[BTU] R.Bott and L.W.Tu, *Differential Forms in Algebraic Topology*, Springer Graduate Texts in Mathematics vol. 82, 1982.

[WEK] C.A.Weibel, *The K-book: an introduction to algebraic K-theory*, Graduate Studies in Math. vol. 145, AMS, 2013.